Compositional Tutorial*

Basic Pitch and Time Relations in Twentieth Century Music

This text is a supplement to my book *The Whistling Blackbird: Essays and Talks on New Music* (Rochester, NY: University of Rochester Press, 2010). It can also be read independently or possibly as an adjunct to my other articles and books, especially to *Composition with Pitch-Classes: A Theory of Compositional Design* (New Haven and London: Yale University Press, 1987). In *The Whistling Blackbird* I endeavored to explain technical matters as they came up in the discussion of musical structure. Such explanations, however, were often simplified or partial. I was and am aware that this may confound or frustrate the reader who wishes to know the whole story, not just the issues at hand. As Popper has said, facts are small theories; so, for instance, to say that this group of notes is a partition, or a transposition, or a set-class, etc. carries with it a theoretic orientation to music structure—and one that I have no desire to hide. Nevertheless, *The Whistling Blackbird* is long book, and there was not enough room to completely explain the theoretic content and context of all technical terms. Hence this extended tutorial.

In this text I present fundamental concepts and principles that can help today's composers write music. Even if many excellent composers are able to make music without these ideas and procedures, a good deal of the concert music of the twentieth century can be addressed, understood, and interrelated if such technical matters are mastered and become part of one's musical practice. Indeed, much of what I will present was invented (or discovered) by progressive composers who wished to refine and extend their compositional craft and by analysts who wanted to understand the structure and form of new music. Today, this parts and aspects of this material is often taught in (graduate) courses that survey recent music repertoires and/or study atonal, twelve-tone, and other related musical theories. In any case, such technical categories such as atonal, twelve-

* I am grateful to James Romig for carefully editing this text; any remaining errors are mine.

tone, serial, post-tonal, neo-tonal, and minimal all partake of these basic ideas in varying degrees.[1]

While many readers may have encountered such material in their own experience and education, others may have spotty or even erroneous knowledge, and still others may be completely mystified and possibly intimidated. Therefore I have tried to present this material as simply as possible with pertinent musical examples. The scraps of music I have composed for the examples are perhaps unduly didactic, and I hope sophisticated composers will forgive me. The music I discuss in *The Whistling Blackbird* provides more nuanced and far-reaching instances of what this theory enables.

Time and pitch spaces

First, I must differentiate the different senses of time and pitch that are too often conflated in musical discourse. Western concert music is often considered (quantatively) as a series of notes in a specific timing. This, of course, is a naive theoretic presumption that—while supported by traditional western notation—hardly describes the experience of music. In part, this is because it ignores most of the attributes of "notes." In addition to its onset time and pitch, a note has a duration, a timbre, and a loudness; moreover, it may not even have a clearly heard pitch (as in a drum stroke) or an exact starting or ending time (when it fades in or out among other notes). Nevertheless, a good deal of music's basic structure is based on the time and pitch of notes.[2] So, as I have said elsewhere in this book, we can make a distinction between the structural functions of time and pitch versus the other attributes. These other attributes, however, do have musical function. Until the beginning of the twentieth century they were articulative—that is, used to point out and group the pitches and time points of music in various ways. New music has "liberated" these attributes so that any aspect of a note may have primary structural significance. But, in addition, pitch and time themselves have been also freed from their traditional moorings. The creation of music that is not tonal and is without meter led to a fresh understanding of pitch and time. It was realized by many composers—and later explicitly stated in theoretical texts—that pitch and musical time have at least three distinct attributes and that these attributes are isomorphic. This means that the abstract models of pitch can also be used to describe musical time. This does not mean that musical time and pitch (can be made to) “sound” the same, but that the structural resources of one are functionally identical to that of the other.[3] This unification of time and space (pitch) in music was an important advance and it forms the basis of this tutorial. We will distinguish three basic “spaces” of pitch among others; these are contour, pitch (per se), and pitch-class. These spaces can be used to describe various pitch functions in common practice concert music. But in that body of music, their interaction is regulated and limited according to both explicit and covert rules. In new music, these pitch spaces can interact in any possible way resulting in various musical “styles” and practices.

Contour space

<Example 1 goes here>

A contour is a series of pitches. The set of all contours is *contour space*, the simplest pitch space. Most anyone can recognize contours and differentiate them from one another. The identification of a contour depends only on the ability to tell whether one pitch is higher than, equal to, or lower than another. In contour space the exact size of the intervals between the pitches does not matter. The pitches can be precisely tuned or not, and might even be indefinitely pitched noises (like the different notes of a drum, or the difference between the vocal sounds “s,” “sh,” and “th” for instance). Example 1 shows six instances of the same contour, each with its own tempo, rhythm, and instrumentation. The example also shows that the contour can be notated by a series of numbers. The notes of the contour are named by the numbers starting with 0 and ascending in order of pitch height. The notation shows the labeled notes in temporal order. The contour in the example, <3,4,2,3,0,1,1> has 5 distinct pitches, labeled from 0 to 4. Pitches 3 and 1 are repeated in the contour.

Pitch space

This is a space of pitches each differing by a minimal interval of some size. The pitch space I will describe is the familiar, equal-tempered set of tones differing by a tempered semitone, each equal to a twelfth of an octave. This set includes all the notes on a musical keyboard. In theory there are an infinite number of pitches in a pitch-space, but human hearing limits this to about 95 pitches. Pitches can be notated according to pitch height by numbers from 0 to the highest note we can hear. However it is more convenient to number middle C as 0 and count up (by half-steps) with positive numbers to the highest note we can hear and down (by half-steps) below 0 with negative numbers to the lowest note we can hear. Thus the piano keyboard's lowest note is -39 and the highest is 48. A more traditional labeling system is the use of the pitch-class and a register. (A *register* is from a C to the next highest B.) So, given that C4 is middle C, the lowest note on the piano is A0, the highest is C7, the lowest note of the cello is C2, A440 is A4, and so forth. Of course, ordinary staff notation is also a perfectly useful way of showing pitch sets and their relations.

<Example 2 goes here>

Chords (or simultaneities) are (unordered) subsets of the pitch-space and the order of notes in a melody is a series of pitches. See Example 2.

<Example 3 goes here>

An interval in pitch space is indicated simply by the number of semitones from one pitch to another. If the interval is ascending it is labeled by plus sign ("+"); if it descends it is labeled with a minus sign ("–"); if the interval is between two simultaneous pitches it is unsigned or (in other texts) given a plus/minus sign ("±"). See Example 3.

The intervals 12, -12, ±12, 24, -24, ±24, 36, etc (octaves, and multiple octaves) have no special meaning or function in pitch-space. However, we are so used to hearing octaves as equivalence intervals that attending to pitch space relations alone (without attending to the pitch-class of notes) is difficult to master. This is an important reason why many composers who are interested in asserting pitch relations without pitch-class function often avoid using octaves in chords or between neighboring pitches.

But whether we attend to octaves as special pitch space intervals or not, it is within pitch and contour space where musical gesture occurs. We hear music going up and down in a linear space. We may notice how the spacing of chords and the voice leading of pitches affect our perception of musical balance and continuity.

Pitch-class space

<Example 4a goes here>

When the beginner is taught that the names of notes start with A, proceeds to G, and then repeat cyclically, he or she is introduced to the notion of pitch-class. For our purposes, a pitch-class is a set of notes related by any number of octaves. But when we say we hear a pitch-class such as Bb we do not necessarily hear all the Bbs at once, but one or more of the Bbs in the pitch-class Bb. Stating that we hear a Bb tells us nothing about how high or low the note is. Therefore pitch-class is not about highs or lows or ascending or descending gestures. We can model pitch-classes by a space of 12 pitch-classes (henceforth pcs) where the pcs are arranged in a circle. See Example 4a. We can notate pitch-classes on a staff, but this can be confusing unless we keep the notes within a pitch register of 11 adjacent pitches. But even this has limitations, for the same pitch class collection will appear differently depending which pitch-class is the lowest note of a register. Therefore theorists use note names such as C# F Bb or the numbers 0 to 11 to show pitch-classes (as shown in Example 4a). The definition as C as pc 0 is arbitrary and used as a convenience; sometimes the pc label “0” is set to some other pitch-class such as D or F#.[4] As the example shows, we label pitch class Bb (A#) as “10” or “A” and pitch-class B natural as “11” or “B.”[5].[6]

<Example 4b goes here>

<Example 4c goes here>

Pitch classes are usually realized, represented, or articulated by pitches in pitch-space. Example 4b shows several different realizations of the C-major chord in pitch. In traditional theory we refer to these different realizations as different “inversions,” “spacings,” and “doublings” of the C-major chord. Example 4c shows some realization of a series of pcs. The example shows that it is perfectly appropriate for a pc to be realized as two or more octave-related pitches in the pitch-space. The first four cases show horizontal (or “melodic”) realizations of the series of pcs in pitch and time; the last case shows the pitch series realized only in pitch (not time). This ordering issue will be discussed later in the tutorial. Note that these various realizations sound quite different, but their pitch class identity may be also heard through the differences. This implies that the realization of pc entities and relations in pitch, in contour, and in different time spaces is an important site for the management of unity and diversity in music.

<Example 4d goes here>

<Example 4e goes here>

<Example 4f goes here>

There are only 12 ordered intervals between the pcs. In this tutorial I use the numbers 0 to 11 (or “B”) for these. Note that the ordered pc interval from pc 3 to 4, an instance of pc interval 1, can be realized as a move from any Eb to any E, so the realization might be an upward or downward pitch interval. See Example 4d. This is so for all of the twelve intervals. When two pitch-classes are simultaneously sounded, the shortest distance around the pc circle from one pc to the other names the interval. So the simultaneous interval from C to Ab (or G#) is labeled 4. Ab is 8 steps from C clockwise around the circle, but only 4 steps counterclockwise. See Example 4e. Such simultaneous intervals are called unordered intervals, or *interval-classes* (abbreviated *ics*) in the theoretical literature. There are only seven ics, from 0 to 6. Example 4f provides some realizations of both the unordered and ordered pc interval 3 in the pitch-space.

In most musical systems, such as tonal or twelve-tone music, the basic syntax of the system is based on a combination of pitch and pc relations; the pc relations provide the basis for the system supplemented by rules that determine how pcs are to be ordered in pitch and time. In this way we see that pc relations often provide unity and coherence, and pitch relations provide diversity and character. However, such a division of labor need not be (entirely) the case in some systems, and in yet others pitch may supervene over pc, or pc relations might be ignored altogether. On the other hand, it would be a greatly impoverished music that depended on pc alone.[7]

Time spaces and sequential time

As with pitch, in time we can distinguish three kinds of spaces: *sequential time*, *measured time*, and *cyclic time*. These three time spaces correspond to the three pitch spaces: sequential time corresponds to contour space; measured time to pitch space; and cyclic time to pitch-class space. While sequential time involves only the ordering of musical entities not their timing or pacing, the other two time spaces are more complicated.

Measured time

<Example 5a goes here>

<Example 5b goes here>

<Example 5c goes here>

Measured time is a space of equally spaced time points, a series of pulses. We can model a measured time space as a series of notes of the same rhythmic value, such as a series of sixteenths. See Example 5a. A time point set is a subset of these time points. Such a set may be heard as rhythm. See Example 5b. Duration is the distance between two time points and is therefore an unordered interval. Ordered intervals are duration distances measured from one time point to another and may be positive or negative respectively corresponding to whether they measure duration from a time point x to a later time point, or to one before x.[8] See Example 5c.

Cyclic time

Cycles of time points form various *cyclic time spaces*. The time points in such spaces are called *cyclic time points* or *beat-classes*, The cycle of 12 time points is isomorphic to the pitch-class space of twelve pcs. Just as pcs are realized in pitch, beat-classes are realized in measured time. This interaction between a cyclic time of 12 beat-classes and measured time was invented by Milton Babbitt and is called the *time point system*.

In *The Whistling Blackbird* I discuss these time spaces in the essay on musical time; therefore I will not elaborate them much further here. However, I will illustrate some relations between and among temporal entities in the examples in this tutorial. I will also show in the examples how other non-pitch attributes of notes may also interact with the three pitch spaces and others to distinguish and highlight their functions.

Transformations

Along with the properties of the time and pitch spaces, I discuss the nature of certain (often well-known) musical transformations. A *transformation*[9] is an action one performs on a musical entity such as a note or a group of notes, or on an entire passage or musical texture.[10] Examples of pitch transformations are transposition, inversion, and the like; however, these do different things in each of the three pitch spaces. Likewise, shifting a musical entity forward or backward in time or retrograding it are examples of temporal transformations, and these are different in the three types of time spaces.

Speaking more generally, transformations can be used to change musical entities in predictable ways. The result of transformation on a musical entity X that creates something completely different from X has limited value (notwithstanding a certain, and possibly desirable, fecundity). Transformations are more generally used to obtain audible relations among different musical entities. Without transformation there is only identity and difference. But music's particularity is pervaded by degrees of similarity resulting in all sorts of interesting complexities and ambiguities. Understanding musical transformations make it possible for us to specify and regulate such multifaceted features.

Musical events that are related by specified transformations can form sets that contain things of the same kind or function.[11] A simple example is (the concept of) a major chord. A major chord is actually not a single entity like a C-major chord (which is already a set of the notes (pitch-classes) C, E, and G); it is rather the collection of chords related by (pitch-class) transposition including the C-major chord. Thus a major chord stands for twelve distinct chords including the Db-major, A-major, F#-major chord, and nine others (not counting enharmonic spellings). When we say that a given chord in a piece of music is a major chord we are actually saying that the chord in question is a member of the class of major chords, or that it is an example or instance of the class of major chords. In traditional theory we infrequently make the distinction between a chord and a class of chords, presumably because that distinction is (thought to be) too trivial to state as such. But this conflation of entity and set can confuse and it masks the fact that sets of entities that "sound" alike or similar are related by transformations such as transposition.[12]

Talk of transformations also allows us to also specify what does not change under a transformation. This is called *invariance*. So when I transform a major scale by transposing it by perfect fifth ("up"), some of the notes in the two scales are the same.[13] We say these notes are *invariant*.[14] Here is the Eb-major scale: Eb F G Ab Bb C D. Here is its transposition "up" a perfect fifth, a Bb-major scale: Bb C D Eb F G A. The notes in common are the invariants: Eb F G Bb C D. Some sets of notes are totally invariant under certain transformations. Sticking with transposition as our example of a transformation, any diminished-seventh chord is totally invariant under transposition "up" a minor third.

{C Eb F# A} is the same chord if we transpose it "up" by a minor third.[15] Thus it is an example of a transpositionally invariant chord.[16]

Lastly, it must be pointed out that transformations such as transposition (in pitch) or retrograde (in time) do (slightly) different things in the different spaces. For example, in the previous paragraph we performed transposition on pitch-class entities in pitch-class space. (The "notes" were pitch-classes). I gave an example of a transpositionally invariant chord. Yet, in the pitch space (not the pitch-class space) it is impossible to construct a chord (of pitches per se) that is completely transpositionally invariant.[17] Therefore, transformations with the same name may do different things in each of the spaces.[18] I shall make this clear below.

Sets

Strictly speaking, sets are collections of things called members or elements. The only thing that matters is the content of the set—the members—which can be displayed in any order. If the set is the English alphabet, A is a member, so is B, so is Q, etc. Theorists often use letters or words to name sets. So we can name the alphabet "ALPHA." Membership can be indicated by the membership sign "∈". So we can write A ∈ ALPHA, B ∈ ALPHA, Q ∈ ALPHA. "∉" means "is not a member." Thus, 9 ∉ ALPHA, and Σ ∉ ALPHA. We can spell out the content of a set by listing its members in curly brackets: "{" and "}". Thus the set of notes {A B C D E F G} are the members of an A-minor scale. We can call this set Z and write "Let Z = {A B C D E F G}". However, notice that Z also contains the notes of the C-major scale, D-Dorian scale, etc. This is because a set only registers content; order does not matter. So {C D E F G A B} and {D A G F B E C} are no more than other ways of defining Z.

We need other types of sets in music, but I want to focus on the use of unordered sets. We often consider simultaneous notes to form an unordered set. This can make sense; but such simultaneity can also be considered an ordering of pcs in pitch (from low to high or vice versa), in which case it is an ordered set. Moreover, a succession of pitches or pcs in time might be a temporal unfolding of an unordered set, as in the way an *arpeggiation* or an *Alberti bass* pattern articulates a chord in tonal music. Thus we can say that an unordered set is "conceptually" unordered, even when its members are presented in order in time. Whether a passage of music is ordered or conceptually unordered is a judgment call for the analyst and composer.

In addition to sets *per se* (also called unordered sets, collections, and/or classes), we have *ordered sets*, *cycles*, and *partially ordered sets*. Here's an example of each of these types. The pitches of a melody form a series; so it is an ordered set. A scale is a cycle of pitch-classes and an ostinato is a cycle of durations between time points. A figured bass notation such as "6 written under an F natural" denotes a partially ordered set. It indicates a set of notes ordered from low to high in pitch so that F is the lowest pitch-class and the pitch-classes F, A and D may occur in any order above the F. See Example 6.

<Example 6 goes here>

We show the members of an ordered set in angle brackets: "<" and ">". Thus, <5 7 8> and <7 5 8> are not the same ordered set. An ordered set may have duplicated members, either adjacently or not. For instance, <5 6 8 8 6> is an ordered set. In twelve-tone music an ordered set of pitch-classes is called a *segment*, and a *row* is an ordering of all twelve pitch-classes without duplication.[19] The terms *series*,[20] *chain*, and *string* are other names for ordered sets.

Some partially ordered sets (also called *posets*) can be notated by a combination of angle and curly brackets.[21] As in Example 6, our figured bass example can be notated as < F {D F A}>. The F comes first (in order from low to high) then followed by the unordered set {D F A}, that is, any order of D, F, and A.

Sets that are included within larger sets are called *subsets*. The larger sets are called *supersets*.

(Unordered) set D = {1246} is a subset of E = {012345678}. We can show this with the symbol "⊂", which means "is included in" or "is a subset of." So we can write $D \subset E$. Note that ⊂ is not the same as ∈, which tells what the members of a set are. The symbol ⊂ indicates that one set is included in another without mentioning the members, if any, in each.

Consider ordered sets F = <0352>, G = <3052> and H = <1043562>. $F \subset H$ but $G \not\subset H$. ("⊄" means not included.). This shows us that both the members and their order must be replicated in the superset set if there is to be inclusion among ordered sets.[22]

The *intersection* of two or more sets is the set of elements that are shared by (or common to) each set. We show intersection by the symbol ∩. So if A = {a b c d} and B = {b d e f}, set C is the intersection set {d b}. So we can write $A \cap B = C$.[23]

The *empty* or *null set* is without members. It is shown by "{}" or the symbol "∅." If two sets G and H have no members in common, their intersection is the null set. We write $G \cap H = \varnothing$ or $G \cap H = \{\}$.

The *union* of two or more sets is the set of elements from all of them taken together. We show union by the symbol ∪. Using sets A and B from the previous paragraph again, their union is D = {a b c d e f}. We can write $A \cup B = D$.

The number of members in a set is its *cardinality*. We use the "#" to show this. #X is the number of members in set X. So we can state using the sets from above: #A = 4, #B = 4, #C = 2, and #D= 6.

Sets may be either finite or infinite. The number of pitches is infinite in principle, but the number of pitch-classes is 12. When the members of sets are selected out of an explicit repertoire, we call that repertoire the *universe of discourse* or *U*. So for example, the pitch-class space is the universe of discourse for pitch-class sets.

Given a set A within U, the members of U not included within A form the *complement* of A symbolized as A′. From this we see that complements are shown by the prime sign ′. The complement of the C-major chord (in the pc space) is the set = {Db D Eb F F# Ab A Bb B}. If we call the notes of the C-major scale W, then W′ = {F# G# A# C# D#}. We can literally see this on the piano keyboard as white versus black keys.

Some simple identities follow:

$A \cup A' = U$ (the union of A and A′ is the universe of discourse.)
$A \cap A' = \varnothing$ (the intersection of A and A′ is the null set.)
$\#A + \#A' = \#U$. (The number of elements of A plus the number of elements of A′ is equal to the number of elements in the universe of discourse

A *partition* of a set X is a collection of non-intersecting subsets of X that exhausts the members of X. If X = {abcdef} a few of its partitions are: { {ab}{cd}{ef} }; { {a} {cef} {bd} };
{ {abcef}{d} }; { {a} {b} {c} {d} {e} {f} }. With respect to U, the simplest partition is a set and its complement. Here is an example in the pitch space: { {012345}{6789AB} }—two non-intersecting chromatic hexachords.[24]

Sets and Transformations in Contour Space

<Example 7 goes here>

<Example 8 goes here>

In contour space a contour is denoted by an ordered set of numbers, as described above in the section on contour space. A given contour may be realized in different ways as shown in Example 1. Many of these realizations are pitch sets that share the same contour. In contour space, a contour may be transformed into another under retrograde, inversion, and retrograde inversion. The set of contours related by these transformations is called a *contour-class*. See Example 7. Other transformations of contour involve permutation, deleting and adding time points at the beginning, end, or in the middle of a given contour. A contour may include other contours as subsets. These subsets may be found contiguously or not within the supercontour.[25] See Example 8. Note that the included contours are interrelated by R, I, RI and therefore form a contour-class.

Sets and Transformations in Pitch Space

Contour preservation

<Example 9 goes here>

Among the various transformations available for use in the pitch space, we can specify transformations that preserve the contour of an ordered set of pitches. Thus contour can be defined as a class of ordered pitches in p-space. An example is shown in Example 9, in which contour expansion preserves contour.

Spacing preservation

<Example 10 goes here>

Another class of transformations preserves the spacing type of an unordered p-space set (henceforth *psets*). The spacing of a pset is a list of the intervals from pitch to pitch from low to high. For instance, given the set {3 4 7 12}, its spacing is <1 3 5>. There are five spacing types. Each of the psets in Example 10 illustrates one of the five *spacing types,* labeled I to V.

> I Spacing is from large to small intervals (pset A). We may call this spacing *overtone spacing,* as the pset A verifies.

II Spacing is from small to large intervals (pset B). We call this *inverse overtone spacing.*

III Spacing is (roughly) *uniform* (pset C).

IV Spacing has wide intervals at extrema, gradually narrowing to small intervals at its center (pset D). This is *focused spacing.*

V Spacing has narrow intervals at extrema, widening to larger intervals at its center (pset E). This is *barbell spacing.*

In the example, the psets F through J are transformations (via various interval expansions and pitch additions or deletions in p space) that preserve the spacing types of A through E respectively.

Interval preservation in pitch

<Example 11 goes here>

When we transpose a pset or ordered pitch set (henceforth, *pseg*) the intervals between the pitches remain the same. We use the symbol T_n to indicate transposition up n pitches in the pitch space. We write the T_n to the left of the pset or pseg to indicate that the set is transposed up n pitches. If pset A = {-3 6 7}, then T_4A or T_4{-3 6 7} is {1 10 11}; T_9A = {6 15 16}. If the n is negative, such as -2, we transpose the pset or pseg down. Thus, T_{-2}A = {-5 4 5}. See Example 11 to see these examples and others on the staff.

<Example 12 goes here>

If we are dealing with a pset, we may specify the intervals within it as unordered, that is, as unsigned numbers. So the unordered intervals between pairs of pitches in the pseg A = {-3 6 7} are 9, 10, and 1.[26] In the case of a pseg like B = <4 6 -2> the intervals are ordered; in this case the intervals are +2, -6, and -8.[27] See Example 12 to see these psegs and intervals written on the staff.

Inversion is the operation that changes the sign of each pitch in a pset or pseg. Inversion is denoted by the symbol “I” written to the left of a set or its name. So using our pset A and pseg B. IA = I{-3 6 7} = {3 -6 -7}; and IB = <-4 -6 2>. I inverts around middle C, or pitch 0.

In the case of a pset, its unordered intervals do not change under inversion. The unordered intervals in pset A are 9, 10, and 1, when we inspect the unordered intervals in IA, we get the same intervals.[28] See Example 12.

In the case of a pseg such as B, the intervals are inverted under inversion. The ordered intervals in B are +2, -6, and -8. In the inversion of B, IB = {-4 -6 2}, the intervals are -2, +6, and +8.[29] See Example 12.

<Example 13 goes here>

Transposition and inversion can be applied together on pitch entities. By convention we do inversion first.[30] We therefore write T_nI to the left of a pitch, pset, or pseg. This means first we do I, then we do T_n. So given A = {-3 6 7}, T_4IA = {7 -2 -3}.[31] See other T_nI transformations in Example 13.

In sum, pitch space transposition preserves intervals, and pitch space inversion (with or without transposition) preserves unordered intervals, but inverts ordered intervals.[32]

Retrograde and other permutations may be performed on psegs; they have no effect on single pitches or psets. The retrograde of C = <8 5 0 -3 7> (written RC) is <7 -3 0 5 8>.

<Example 14 goes here>

Classes of psets and psegs

Psets may be gathered into set classes under the operations of T_n and T_nI. A *set-class* contains all the psets that are related by transposition and/or inversion. It is therefore a set of sets. The motivation for such a grouping of psets is that ordered intervals are preserved in all the members of a set-class. Example 14 shows some of the psets in the set-class containing the pset K = {0 2 13} both on the staff and in number notation.[33] We write SC(X) to denote the set-class that contains the pset X.[34] So Example 14 shows psets in SC(K) or SC({0 2 13}).

We can define pseg-classes as well. These include all the T_n, and/or I, and/or R transformations of a given pseg.

Other transformations in pitch-space

There are of course many other transformations on sets in pitch space. One can produce subsets by deleting pitches, supersets by adding pitches; one can permute psegs, construct new sets from others sets via intersection, union, etc. In this way, just a few pitch sets can generate a galaxy of psets and psegs. Whether these relations can be heard will depend on the way the pitches are articulated in the time spaces and by articulators such as dynamics and timbre.

Sets and Transformations in Measured time

As mentioned earlier, a measured time space is an infinite set of time-points, each leading to the next by a certain unit of time. We can show a measured time space by writing a series of sixteenth notes as in Example 5a. Measured time sets are finite subsets of the space. Example 5b shows a few such sets, which form various rhythms. We label the time points of a measured time space with consecutive numbers. For convenience we can name some convenient central time point 0 to accommodate the transformation of inversion. As stated earlier, an unordered interval or duration between two time points is the number of time points from either one to the other. Ordered intervals are the durations from one time point to another and may be positive or negative respectively corresponding to whether they go from a time point x to a later time point, or to one before x. Please review Example 5c.

<Example 15 goes here>

Two familiar transformations on time points sets are *shift* and *retrograde,* which may be performed alone or in concert. These two transformations are exactly analogous to the transformations of transposition and inversion in pitch space. This is because measured time and pitch space are isomorphic. So we could designate a shift of a time point set n time points later by T_n, and designate retrograde by I. But we shall use S_n for a shift of n time points ahead and R for retrograde. An inspection of Example 15 will show that S_n and R both preserve unordered intervals, while S_n preserves ordered intervals and R retrogrades them.

<Example 16 goes here>

We can construct set-classes of *time point sets*, or *rhythms*, such that rhythms of time points that are related by S_n and/or R are in the same time point set class. See Example 16.

<Example 17 goes here>

Time point sets can be articulated by any music events that start (or end) at a member of the set. The time point set in Example 16 Q = {2 3 4 5 6 7 8 9 10 11 12 13}, which is simply the series of time points from 2 to 13, can be constructed from subsets that are members of the set-class containing the time point set P = {2 3 5}. As shown, the members of this set class that "construct" Q are P = {2 3 5}, S_9RP = {4 6 7}, S_6P = {8 9 11} and S_{15}RP ={10 12 13}. This construction is a partition of Q and can be musically communicated by using a different articulator for each subset. Example 17 shows a few different ways this can be done. In this way, the articulators “analyze” Q into the subsets that generate it.

Because measured time is isomorphic to pitch space, there is a transformation in measured time that corresponds to preserving spacing in pitch space. We will say that two time sets are *D-related* if the sizes of their respective successive durations form the same pattern. Here are two time point sets in the D-relation: X = {0 3 10 12} and Y = {2 7 13 17}. The sequence of successive durations of X is <3 7 2> and of Y is <5 6 4>. In each of the duration sequences the shortest duration is last, the next shortest duration is first and the longest duration is second. Coding the lengths of durations in each by the sequence 0 1 2, the permutation 1 2 0 describes the D-relation between the two sets. This is analogous to having the same spacing among pitch sets. We can define D-set types where the D pattern is 0, 1, 2 etc. so that the durations get larger as we progress though the set. This is analogous to type II (inverse overtone) spacing in the pitch space. The effect is one of *ritardando*. The reader is encouraged to define what the other four spacing types would correspond to in measured time, and construct measured time sets that are members of each type.[35]

Sets and Transformations in Pitch-class Space

In this section, I will follow the sequence of topics in previous exposition of the pitch space. Much of the formalism will correspond, but the examples will be different. The complication here is that pc relations must be mediated via pitch and various times spaces. This is a very important point since relationships among pcs and their unordered and ordered sets can be either highlighted or disguised (even hidden) by the way they are articulated.[36] (I will provide some examples below.) This means there is no point in using pc relations in music if one does not attend to how they can be expressed in pitch and other spaces[37]—unless one wishes to write music where pcs are organized according to musical relations that are nevertheless not manifested in the music.

In the following a *pcset* is an unordered set of pitch classes. A *pcseg* is an ordered set of pcs.

Pcsets

<Example 18 goes here>

Pcsets are unordered sets, but because they contain pcs, they must be expressed or articulated in both time and pitch. Even if the content of a pcset is played simultaneously, it must be ordered in pitch. So a pcset {348} could be played as any number of pitch space chords, with any of its pcs as the lowest, highest, or middle notes in pitch and in any region of pitch space. When the pcs in a pcset are to be ordered in time, there are two orderings afoot; a temporal (linear) ordering and a pitch space (vertical) ordering. These two orderings express the pcset as a contour. See Example 18.[38] As I mentioned above, even though the pcset is ordered in two dimensions, it is conceptually unordered. This means that the orderings used to express the pcset are not as significant as the content of the pcset. Nevertheless, such orderings in pitch and time might be heard as significant to a listener or analyst, so the passage at hand might be interpreted as one or more ordered sets or pcsegs.

Interval preservation

When we transpose a pcset or pcseg, the intervals between the pcs remain the same. We use the symbol T_n to indicate transposition. The n indicates that we add n to each member of the pcset or pcseg. When a pc is changed in transposition so it would exceed 11 (or B), we take 12 from the result. This is called mod-12 arithmetic. So $T_7 6$ (or $T_6 7$) = 6 + 7 in the pc space is 1 (13 – 12). We do this because the pcs form a circle with 0 following 11 clockwise.

<Example 19 goes here>

We write the T_n to the left of the pcset or pcseg to indicate that the set is transposed by n. If pcset A = {9 6 7}, then T_4A or T_4{9 6 7} = {1 10 11}; T_9A = {6 3 4}.[39] See Example 19 to see these examples and others realized in different ways in pitch space and time.

If we are dealing with a pcset, we specify the intervals within it as unordered, from 0 to 6.[40] So the unordered intervals (or ics) between pairs of pc in the pcseg A = {9 6 7} are 3, 2, and 1.[41] In the case of a pcseg like B = <4 6 10> the intervals are ordered, their sizes varying from 0 to 11. In B the intervals are 2, 6, and 4.[42]

<Example 20 goes here>

Inversion is the operation that changes a pc into its inverse. The inverse of a pc x is 12-x. A pc and its inverse add up to 12 (or 0 in mod twelve arithmetic). A table of pcs and their inverses is presented in Example 20.

Inversion is denoted by the symbol "I" written to the left of a pcset or pcseg or its name. So using our pcset A and pcseg B. IA = I{9 6 7} = {3 6 5}; and IB = <8 6 2>.

<Example 21 goes here>

In the case of a pcset, its unordered intervals do not change under inversion. The unordered intervals in A are 3, 2, and 1; when we inspect the unordered intervals in IA, we obtain the same intervals.[43] See Example 21.

In the case of a pcseg such as B, the ordered intervals are inverted under inversion. The ordered intervals in B are 2, 6, and 4. In the inversion of B, IB = <8 6 2>, the ordered intervals are[44] 10, 6, and 8. See Example 21.

<Example 22 goes here>

Transposition and inversion can be applied together on pc entities. By convention we do inversion first.[45] We therefore write T_nI to the left of a pitch, pc, or pseg. This means first we do I, then we do T_n. So given A = {9 6 7}, T_4IA = { 7 A 9}.[46] See other T_nI transformations in Example 22.

In sum, pc space transposition preserves intervals, pc space inversion (with or without transposition) preserves unordered intervals, but inverts ordered intervals.[47] As we see below, pcsets related by T_n and/or I have the same interval-class vector (ICV).

Retrograde and other permutations may be preformed on pcsegs; they have no effect single pitches or pcsets. The retrograde of pcseg C = <8 5 0 9 7> (written RC) is <7 9 0 5 8>.

<Example 23 goes here>

Before continuing further, in Example 23 I will show three pitch realizations of T_n- and T_nI-related pcsets. These realizations illustrate how sets in pc space can be made to appear related in ways that seem to contradict their pitch-class space relations. In pc space, X is related to T_AIX by transposition following inversion and to T_9X via transposition. But their pitch space realization associates X to T_AIX via a "rough" transposition, and X and T_AIX appear to be related by a "rough" inversion to T_9X. One could also consider the three to be members of the contour-class <01234>, with X and T_AIX related to T_9X by contour inversion (or retrograde). This shows us that we can highlight or camouflage pc relations by the way we realize sets in pitch and contour space. This is not a "problem" with pc space relations if one attends to realization of pc entities in pitch and time, for it allows related pc entities to be musically associated on a continuum from identity to difference.[48]

The transformations on sets in pitch space can be applied to sets in pc space as well. One can produce new sets by deleting pcs or adding pcs, by permuting pcsegs, and by constructing new sets from others sets via intersection, union, etc.

The Interval content of pcsets

The “sound” of an unordered pitch or pcset depends on the intervals within it. In the case of pcsets, we can display the unordered intervals between pairs of its pcs by a notation called an *interval-class vector* (abbreviated *ICV*).[49] The ICV gives the interval-class content for any pcset. Seven successive numbers occur within brackets. The leftmost number gives the number of interval-classes of size 0 (and thus, the cardinality of the SC's members), the second number from the left gives the number of interval-classes of size 1, and so forth until we get to the last (seventh, rightmost) number, which indicates the amount of ic6s in any set within the set-class.[50] For example, [3100110] is the ICV of the pcset {156}. It indicates that this pcset has 3 pcs, one ic1, one ic2, no ic3, one ic4, one ic5 and no ic6. I will return to ICVs in the section on pc space set classes below.

Ordered interval content of pcsegs

While we can register all the ordered intervals between pairs of pcs in a pcseg, we are usually only interested in their adjacent intervals, that is, the ordered intervals between successive pcs. To list these intervals we use the function INT. We write INT(W) to list the adjacent ordered intervals as they appear in order in pcseg W. Let W = <6,2,8,11,2,2,5>. The INT(W) = <8,6,3,3,0,3>.[51] The INT of a pcseg does not change when the pcseg is transposed. The INT is inverted when the pcseg is inverted.[52]

Pc space set-classes

Because the pc space is finite, having 12 pcs, there are exactly 4096 pcsets. This includes U, the set of all pcs, often called the *aggregate,* as well as the *null set* ({} or ∅). Then there are the 12 pcs, the 66 distinct unordered pairs of pcs, the 220 distinct trios of pcs, etc.

Pcsets may be gathered into set classes under the operations of T_n and T_nI. A pc *set class* contains all the pcsets that are related by transposition and/or inversion. It is therefore a set of sets. The cardinality of a set-class—that is, the number of pcsets in the set-class—can vary from 2 to 24 pcsets.[53] The motivation for grouping pcsets into set-classes is that unordered intervals are preserved in all the members of a set-class. In other words, each member of a set class has the same ICV. Thus we may associate an ICV with a set class.

<Example 24 goes here>

Example 24 shows all 24 psets in the set-class containing the pcset {0 1 3} in pc number notation; its ICV is [3111000]. Another set-class containing pcset {0 1 4 5 8 9} is also shown in the example; it contains only four pcsets and its ICV is [6303630]. We write SC(X) to denote the set-class that contains the pset X.[54]

The set-classes in pc space have been enumerated and studied by many composers and theorists. As stated above, a set-class (or SC) is a collection of all the pcsets that are related under transposition (T_n), inversion (I), or both (in pc-space). All of the pcsets within a set-class are of the same cardinality and have the same ICV. In my work I use an adaptation of the list of set-classes in Allen Forte[[55]] and John Rahn[[56]]. (My list of set-classes is found in Appendix B of *The Whistling Blackbird*, or Appendix One of *Composition with Pitch-Classes*, or on my website.)

As indicated in Example 24, a SC's "name" consists of two numbers separated by a hyphen in parentheses followed by its "prime form" written in brackets. The second of the hyphenated numbers indicates the position of a particular set-class on the list and the first number gives the set-class's members' cardinality. The "prime form" portion of the SC's name is simply one of the pcsets included in the current set-class. For example, SC(4-5)[0126] is the name of the set-class whose cardinality is 4, and is fifth on the list of set-classes of that cardinality. [0126], its prime form, is an ordering of one member of the SC. Two SCs, one possessing the complements of the pcsets in the other, have the same number after the hyphen in their names. The reader should consult a set-class table for more information.

<Example 25 goes here>

The chart in Example 25 gives the number of pcsets for a given cardinality and set-classes for each cardinality of pcset. We see that there are 220 three-pc pcsets grouped into 12 trichordal set-classes, 495 four-pc pcsets grouped into 29 tetrachordal set-classes, etc.

The pcsets in a set-class share many of the same properties. They may also "sound"[57] similar since they share the same ICV. Pcsets from different set-classes have different "sounds" and properties. Below I give a few examples.

The set-class SC(4-15)[0146] has the following ICV: [4111111]. This means each of its pcsets have one of each non-zero interval-class. This is why the pcsets in this SC are called all-interval tetrachords. But SC(4-29)[0137] also has the same ICV as SC(4-15). Thus it also contains all-interval tetrachords. But the tetrachords in SC(4-15) are not related to the tetrachords in SC(4-29) by T_n and/or I. If they were, the two set classes would be one. We say that any pcset from one and any from the other are *Z-related*. That is, the two pcsets have the same ICV, but are not related by T_n and/or I.

Other pcsets and set-classes have ICVs that leave out some ics and favor others. The ICV of pcsets in SC(6-7)[012678] is [6420243]. There are no ic3s and many ic1s, 5s, and 6s.

Some SCs have fewer than 24 members. Pcsets within a SC are related by the 24 T_n and T_nI transformations. But if a member of the SC is invariant under one or more of these transformations the SCs contains fewer than 24 pcsets. The SC(6-7) just cited has only 6 pcsets. Taking the prime form {012678} we see that it is invariant under T_6, T_2I, and T_8I.[58] It is also (trivially) invariant under T_0. Thus {012678} is invariant under 4 transformations. Each distinct pcset in the set-class also has this property, so there are

24/4 or 6 pcsets in the set-class.[59] For further clarification, the reader may look at Example 24 to see the invariances of pcsets in SC(6-20)[014589] spelled out.

A given SC's sets may have many other special properties of various kinds. I have taken advantage of many of these properties in my music, as I explain in *The Whistling Blackbird*. A close examination of my SC table is a way to begin learning of some of the special properties of the pcsets in SCs.

When identifying and hearing the set-class of a three-note pcset or pcseg becomes second nature, many pitch relationships in 20th century music become transparent and "natural." Going further to identify the tetrachords and the all-combinatorial hexachords as to their SC affiliation will open this music even further. Nonetheless, it takes some time to develop such skills since sets of pcs can be realized in many different ways in pitch and time. But once one has become familiar with the more typical ways this is done, things become easier and the theoretical points and issues in the tutorial become obvious.

Pc space pcseg-classes

We can define *pcseg-classes* just as we define set-classes. These include all the T_n, and/or I, and/or R transformations of a given pcseg. There are 48 transformations—the 12 T_ns, the 12 T_nIs, the 12 RT_ns, and the 12 RT_nIs. The pcsegs within a pseg-class have INTs related by identity, I, R, and IR. The familiar complex of rows related under these 48 transforms is a pc segment class.[60] This complex is also known as a *row-class*. If a row has a special invariance (such as $P = RT_6P$), then the row class will have fewer than 48 distinct members.

General invariance principle

The last topic of this tutorial might be found in much more advanced texts in (mathematical) music theory, but is it so important and general that I have included it here. This general invariance principle shows how internal relationships within a set can forge relationships between sets. It therefore identifies an "organic" or "generative" principle for building both local and global references in music. The internal structure of a set becomes the way different sets are related. The principle is at the heart of my compositional concerns, and it can be found in the early twelve-tone compositions of Schoenberg, Webern, and Berg. Nevertheless, it is not just a twelve-tone "technique," it has application to any music whose structure involves transformation. I will illustrate the principle with many examples. Examples (1) and (2) elucidate the principle; (3) and (4) are special cases.

(1) Let S contain subsets X and Y.[61]

S can be can be any kind of set (unordered, ordered, etc.) in any of the pitch or time spaces; it can be a row, time point set, contour, and so forth. The subsets can be selected out of S in any way, and they do not have to be same kind of set as S. So one might pick pcs out of an ordered set, and consider them to be unordered.

(2) If FX =Y, then both S and FS contain Y.

F is a transformation. It operates on one of the subsets of S, X and the result is the other subset Y. Thus the subsets of S have a relation under the transformation F. If we transform S itself under F, we get a new set called FS. Let us see what happens to the subsets of S when S is changed into FS. X is changed into FX, and Y is changed into FY. But we said FX *is* Y. Since S contains Y as stated in (1), and FX is Y as stated in (2) and FX is found in FS, both S and FS contain Y.

So what started out as an internal relation within a set (S) in (1), generates an external relation between two sets (S and FS) in (2).

<u>Example in pitch space</u>

<Example 26 goes here>

S is the pseg <0 2 3 4 6 7>, X = <0 2 3>, Y = <4 6 7>. F is the p space transposition T_4. FX = T_4<0 2 3> = Y = <4 6 7>. T_4S = <4 6 7 8 10 11>. Y is both in S (as its last three pitches) and in T_4Y (as its first three pitches). See Example 26.

Example in pc space

S is the pcset {0123678}. X and Y are also pcsets. X = {016} and Y = {238}; both are included in S as stipulated in (1). Y = T_2X as stipulated in (2). T_2S = {234589A}.[62] Y is included in both S and T_2S. Y is shown underlined in S and T_2S below.

S = {0123678}; T_2S = {234589A}

<Example 27 goes here>

Example 27 gives a realization of this situation in pitch space. The pcs of Y are brought out by duration; there are 6 times longer than the other notes in S and T_2S. Note also how the pcs of Y are configured to form mutually retrograded contours but with different pitch realizations. Notice also how the pcs included in S and T_2S are configured in pitch space so to disguise their T_2 relation in pc space.

Example in measured time.

<Example 28 goes here>

S is the unordered time point set {-8 -4 -1 1 2 5 8 10}. X = {-4 1 5} Y = {-8 2 10}. F is the measured time transformation "times 2" called M_2. M_2 augments the rhythm of S so it is now "twice as slow" as before. Y = M_2X so that Y is found within both S and M_2S as shown in Example 28. The two rhythms intersect at the time points of Y; the two bassoons playing this example have the same pitches at the time points of Y to highlight the rhythmic connection between S and M_2S.

(3) As in (2), but X = Y. Then both S and FS share Y (or X).

Since FX = Y, and X = Y, then X (or Y) is invariant under F, that is FX = X (or FY = Y). Note that while X (or Y) remains unaltered under F, S and FS are not the same.

Example in contour space

<Example 29 goes here>

S is the contour <6024153>. See Example 29. In this case, we are interested in preserving the highpoints of S (contour pitches 5, 4, and 6) while transforming S into other contours. We invent the contour pitch operation Z that sends contour pitch 0 to 1, contour pitch 1 to 3, 3 to 2 and 2 to 0. The other contour pitches remain unchanged. So we can show Z as the permutation cycles (0132)(4)(5)(6). Y is the set of contour pitches {456}, since these pitches are not changed under Z, ZY = Y.

Now we apply Z to S.

S = <6024153>; ZS = <6104352>. ZS has its highest pitches at the same order positions as S.

We can apply Z to ZS to get Z^2S = <6314250>.[63] And we can do Z to Z^2S.can produce yet another contours Z^3S = <6234051>. Another application of Z returns us to S ($Z^4S = S$). All four contours are illustrated in Example 29.

Example in pc space.

Here S is a pseg of nine pcs. S = <05A2618B7>[64]

Y = {0127} is the set of pcs at order positions 0, 3, 5, and 8.[65] Y is invariant under T_2I. Therefore T_2IS has the pcs of Y in its order positions 0, 3, 5, and 8. Note that S is a pcseg, but Y is a pcset.

We can find another set of pcs in S that can function in the same way as Y. To avoid confusion I will call this set Y*. Y* is at the order positions 1, 2, 4, 6, 7 of S. Y* = <568AB>. Y* is invariant under T_4I. Therefore T_4IS has the pcs of Y in its order positions 1, 2, 4, 6, 7.

The order position that hold Y and Y* do not intersect and exhaust all order positions of S.[66]

The following diagram shows S, T_2S and T_4IS. The pcs of Y and T_2IY are underlined and the pcs of Y* and T_4IY* bold.

T2IS:	2	9	4	0	8	1	6	3	7
S:	0	**5**	**A**	2	**6**	1	**8**	**B**	7
T4IS	4	**B**	**6**	2	**A**	3	**8**	**5**	9

Example 30 provides a pitch space realization of these three pcsegs.

<Example 30 goes here>

(4) As in (2), but FY = X. Then S and FS share both X and Y.

Note FFS = S, FFX = X, and FFY = Y. F is an involution.[67]

Example in pc space.

Let S be the pcseg <02346>, X = <034> and Y = <236>.

We show the subsets X and Y in S as follows:

```
X: 0   3   4
S: 0 2 3 6 4
Y:   2 3 6
```

Both X and Y are symmetrically arranged about the pc 3 in the center of S, with X on the outside and Y on the inside.

F is RT_6I, and $RT_6IX = Y$ and $RT_6IY = X$. (RT_6I is an involution since $RT_6IRT_6I = T_0$.)

FS = <20346>

```
RT6IX: 2   3   6 = Y = FX
RT6IS: 2 0 3 4 6 = FS
RT6IY:   0 3 4   = X = FY
```

In RT_6IS X and Y are still symmetrically arranged about the central pc 3, but with X on the inside and Y on the outside. See example 31.

Coda

When the reader is comfortable with the musical concepts and notations presented and illustrated in this tutorial, I trust that the technical material and discussions in *The Whistling Blackbird* and other writings by myself and others will become transparent.

[1] Recent music theoretical literature bears this out. For instance, a Stravinskian rotational array can be used to make music by composers of many different compositional

inclinations; and, as a structure, this kind of array can model the phase-pattern music of Steve Reich or bring deeper properties of scales and rhythmic cycles to light.

[2] Of course, such structural models hardly capture the flow of music. Indeed even the idea of "note" may not apply to certain kinds of new music, as in the textural music of composers like Ligeti and Penderecki or a good deal of electronic and computer music by Stockhausen or Chowning. Notes tend to reify music into quanta, when we "actually" hear music as process involving both continuity and disjunction. Even so, music notation and structural thinking is useful to specify what needs to be done in composition and performance to make music flow.

[3] Since pitch and time spaces have different phenomenologies, the structural isomorphism will not sound or feel the same, or even similar, in corresponding time spaces. For instance, while music can go up and down in pitch, it be only experienced as "moving" into the future in time. We need the assistance of memory to let music "move" into the past. Or consider the pitch space and the measured time space; these have the same structure, but there are only about 95 pitches to be heard, while the number of time points is indefinitely large.

[4] In this way, we have another instance of the distinction between fixed do and moveable do solfege systems.

[5] Other labels are found in the literature. Pitch-class A# or Bb is labeled by "t" or "T" and pitch-class B-natural is labeled by "e" or "E." For technical reasons, I prefer the hexadecimal digits "A" and "B" for the pcs 10 and 11, respectively.

[6] In the various texts in this book, I use notes on the staff, letter names, and pc numbers to represent pcs. Only in technical music theory discourse is the use of pc numbers (or integers, mod 12) indispensable.

[7] Such music could be implemented with Shepherd tones, electronically produced complexes of frequencies that are identical with respect to *both* amplitude and pitch when transposed up or down an octave.

[8] There is nothing counterintuitive about a negative time interval. The time interval of +5 between time points x and y says that y occurs 5 time units after x. If the interval between x and y is -5, then y occurs 5 time units before x.

[9] In music theory and some branches of mathematics a transformation is a one-to-one mapping of one domain into another. This means that for every transformation T, there is an inverse transformation T^{-1} that "undoes" T. (T followed by T^{-1} or vice versa does nothing.) Transformations can form mathematical groups that can model symmetry in music. Nevertheless, in the context of this tutorial, "transformation" is used to mean any change to an entity or a domain.

[10] Thinking of musical structure as fundamentally transformational is an important development in recent music theory. More than anyone else, David Lewin introduced transformational concepts into music theory, but many twentieth century composers had already explicitly stated that their music was transformational or metamorphic and many theorists writing on non-tonal music topics had made the distinction between entities and their transformations long before Lewin's seminal book, *Generalized Musical Intervals and Transformations* was published in 1987. Transformations allow one to think of music as being intentional in the philosophical sense, that music is about something. It or someone is doing something—acting—in a piece (for a particular reason).

Transformational thinking also allows us to configure music as process, not just as a string of (usually) abstract entities.

[11] These sets are called *equivalence classes* or *similarity classes*, depending on the kind of transformation that relates their members.

[12] Modern music theory distinguishes among identity, equivalence, and similarity. Here are the definitions of each. (1) Identity: A copy of an entity is identical to it. (2) Equivalence: Two entities are equivalent if they are related by a transformation K so that: K relates an entity X to itself (reflexivity); if entity X is K-related to Y, then Y is K-related to X (symmetry); if X is K-related to Y, and Y is K-related to Z, then X is K-related to Z (transitivity). (3) Similarity: This is equivalence, but without transitivity: So even if X is K-related to Y, and Y is K-related to Z, X is not necessarily K-related to Z.

[13] The reader may notice that I have put the word "up" in quotes in the examples of transpositional invariance. This is to imply that we are implicitly working within the pc-space; the C-major chord, the Eb-major scale and their transpositions are entities within pc space in which there is no up or down.

[14] Now, of course, all the notes of an entity change under a transformation, but the invariant notes are those shared by the entity and its transformation. So the note D in a C-major scale changes to the note A in the transformation of the scale up a fifth. (And the note F in the C-major scale changes to the note C in the G-major scale.) Now both the C- and G major scales contain the notes C and A so these are among the invariants. The notes F in the C-major scale and the F# in the G-major scale are sole non-invariant pitches.

[15] Under the transposition by a minor third, C becomes Eb, Eb becomes Gb (= F#), F# becomes A, and A becomes C. Note that the notion of enharmonic pitches (which presumes equal temperament) must be invoked in order to produce the total invariance. In a system not employing equal temperament, only the notes C, Eb and A are invariant.

[16] Messaien called such chords and scales, *modes of limited transposition*.

[17] The highest note of a chord in the pitch space cannot transpose up into the same chord; and the lowest note of a pitch space chord cannot transpose down into the same chord.

[18] The reason for this terminological confusion stems from previous tonal theory that did not adequately discriminate between contour, pitch, and pitch-class.

[19] In some of the literature, our row is called a *set*, and our unordered set is called a *collection*. This is another unfortunate terminological situation in music theory.

[20] The term series is sometimes used to denote our row.

[21] The angle brackets denote ordered set content, the curly brackets denote unordered unordered set content; this implies that a poset is a combination of ordered and unordered elements.

[22] Inclusion among cycles is no more exclusive than among unordered sets. Cycles (ABC) (ACB) (ACD) and (DCA) are all included in the supercycle (ABCD). This is because any member of a cycle both precedes and follows any member of the same cycle.

[23] Intersection is implicitly used in traditional music theory in common chord modulations and in measuring the "distance" between different keys.

[24] To summarize, given the set X = {abced}: {{ab}{cd}} is not a partition of A since it does not exhaust the elements of X; {{abc}{cde}} is not a partition of X since the two

subsets in the partition intersect in element c. Here are a few (true) partitions of X: {{abcd}{e}}, {{acd}{be}}, {{c}{abde}}, and {{ae}{bcd}}.
[25] A supercontour is like a superset; it is a contour that contains (sub)contours.
[26] 9 (between -3 and 6), 10 (between -3 and 7) and 1 (between 6 and 7).
[27] +2 (from 4 to 6), -6 (from 4 to -2) and -8 (from 6 to -2).
[28] 9 (between 3 and -6), 10 (between 3 and -7), and 1 (between -6 and -7).
[29] -2 (from -4 to -6), +6 (from -4 to 2), and +8 (from -6 to 2).
[30] This is because doing pitch transposition following pitch inversion is not equivalent to doing pitch inversion following pitch transposition. We can indicate the situation by writing $T_nI \neq IT_n$. We say that T_n and I do not commute.
[31] First we do I to A and get {3 -6 -7}, then we do T_4 to that to get {7 -2 -3} or T_4IA.
[32] We can go further to show that a pitch and its T_nI transformation add up to n; that is, a + T_nIa = n. Also, a and T_nIa invert around—are symmetrically disposed around—the pitch n/2. However, if n is odd, this "axis of inversion" is a quartertone between two pitches. The reader can verify these facts by inspecting the cases in Example 14.
[33] A set-class in pitch space is an infinite set, although we only can hear some of its members.
[34] Note that $SC(T_nIX) = SC(T_nX) = SC(X)$ for all n.
[35] For instance, spacing type III (uniform) would entail times sets where the successive duration are all of about the same length: such as <5 5 5 6 5>
[36] There are even cases where a pc relation cannot be articulated in pitch.
[37] Traditional tonality solves this issue so that pitch classes are articulated as pitches according to rules and practices. For instance, adjacency in pitch-class is to be articulated by adjacency in pitch except in special explicit cases involving octave transfer and voice exchange. But such adjacency is also regulated by avoiding voice crossing in pitch space except in special cases.
[38] Vertical ordering of pcsets is often overlooked in analysis of 20th century music; in many pieces, vertical ordering are more structurally important than linear orderings.
[39] A = {9 6 7}, and T_4A = {9+4 6+4 7+4} = {1 10 11}; T9A = {9+9 6+9 7+9} = {6 3 4}.
[40] Remember that an unordered interval or ic between pcs x and y is the shortest distance around the pc circle from x to y.
[41] 3 (from 9 to 6 counterclockwise around the pc circle), 2 (from 9 to 7 counterclockwise around the pc circle) and 1 (from 6 to 7 clockwise around the pc circle).
[42] 2 (from 4 to 6), 6 (from 4 to 10) and 4 (from 6 to 10).
[43] 3 (from 3 to 6 clockwise around the pc circle), 2 (from 3 to 5 clockwise around the pc circle), and 1 (from 6 to 5 counterclockwise around the pc circle).
[44] 10 (from 8 to 6, clockwise), 6 (from 8 to 2, clockwise), and 8 (from 6 to 2, clockwise).
[45] This is because doing pc transposition following pc inversion is not equivalent to doing pc inversion following pc transposition. We can indicate the situation by writing $T_nI \neq IT_n$. As in pitch space, T_n and I do not commute.
[46] First we do I to A and get {3 6 5}, then we do T_4 to that to get {7 10 9} or T_4IA.
[47] We can go further to show that a pc and its T_nI transformation add up to n. mod 12; that is, a + T_nIa = n.
[48] Since pc sets related by T_n and/or I are members of the same set-class, the statement that two musical "things" are instances of members of the same set-class (SC) does not

tell us the "things" necessarily sound alike. For the composer, such a statement says that the two instances *could* be made to sound alike or not, based on how their pcs are realized in time and pitch. For the analyst, the statement suggests relation, but only further description of the time and pitch relations between the "things" will confirm or call in question their sonic similitude.

[49] Note that the term *interval-class* is equivalent to the term *unordered interval*. As stated above, there are seven interval-classes.

[50] Allan Forte uses a six-place ICV he calls an "interval vector," which is the same our seven-place ICV, but without the first place. So, for our ICV [3210000], Forte's interval vector is [210000]. An advantage of our ICV is that its first place gives the cardinality of the pcset associated with the ICV.

[51] Note that for any pcseg X, the INT(X) is also an ordered set. #(INT(X)) = #X – 1.

[52] More formally, for any pcseg X: INT(X) = INT(T_nX); and INT(T_nIX) = I(INT(X)). The following relationships among INTs are perhaps initially counterintuitive. INT(RT_nIX) = R(INT(X)) and INT(RT_nX) = RI(INT(X)). This means, for instance, that the retrograde inversion of a pcseg has the intervals of the original pcseg in reverse order.

[53] There can be no more than 24 pcsets in a set-class, since there are only 24 transformations of an unordered set, the 12 transpositions and their inversions. As I will show later, there can be less than 24 if there are transformations (except for T_0) of a pcset in the set-class that keep the pcset invariant.

[54] Note that SC(T_nIX) = SC(T_nX) = SC(X) for all n.

[55]Forte, Allen. *The Structure of Atonal Music,* New Haven: Yale University Press, 1973.

[56]Rahn, John. *Basic Atonal Theory,* New York: Longman, 1980.

[57] I put the word sound in quotations because when the pcsets from a set-class are realized in pitch and time, their different configurations will have an effect on the sonically similarity of the realizations. We saw this above in Example 23.

[58] Let {012678} = S. T_6S = {678012}. T_2IS = {210876} and T_8IS = {876210}. Since S in unordered all these transformations leave S invariant.

[59] In other words, of the 24 possible operations on pcsets used to define SC membership, four of them are associated with a single pcset. Therefore there must only be 6 distinct pcsets in the SC.

[60] There is no "official" list of pseg-classes since the number of psegs is infinite.

[61] Neither X nor Y nor (X ∪ Y) need exhuast X.

[62] "A" in {234589A} is the pc 10.

[63] Z^2 is shorthand for ZZ or doing Z twice. Likewise Z^n means do Z n times. Due to definition of Z, Z^4 = Z.

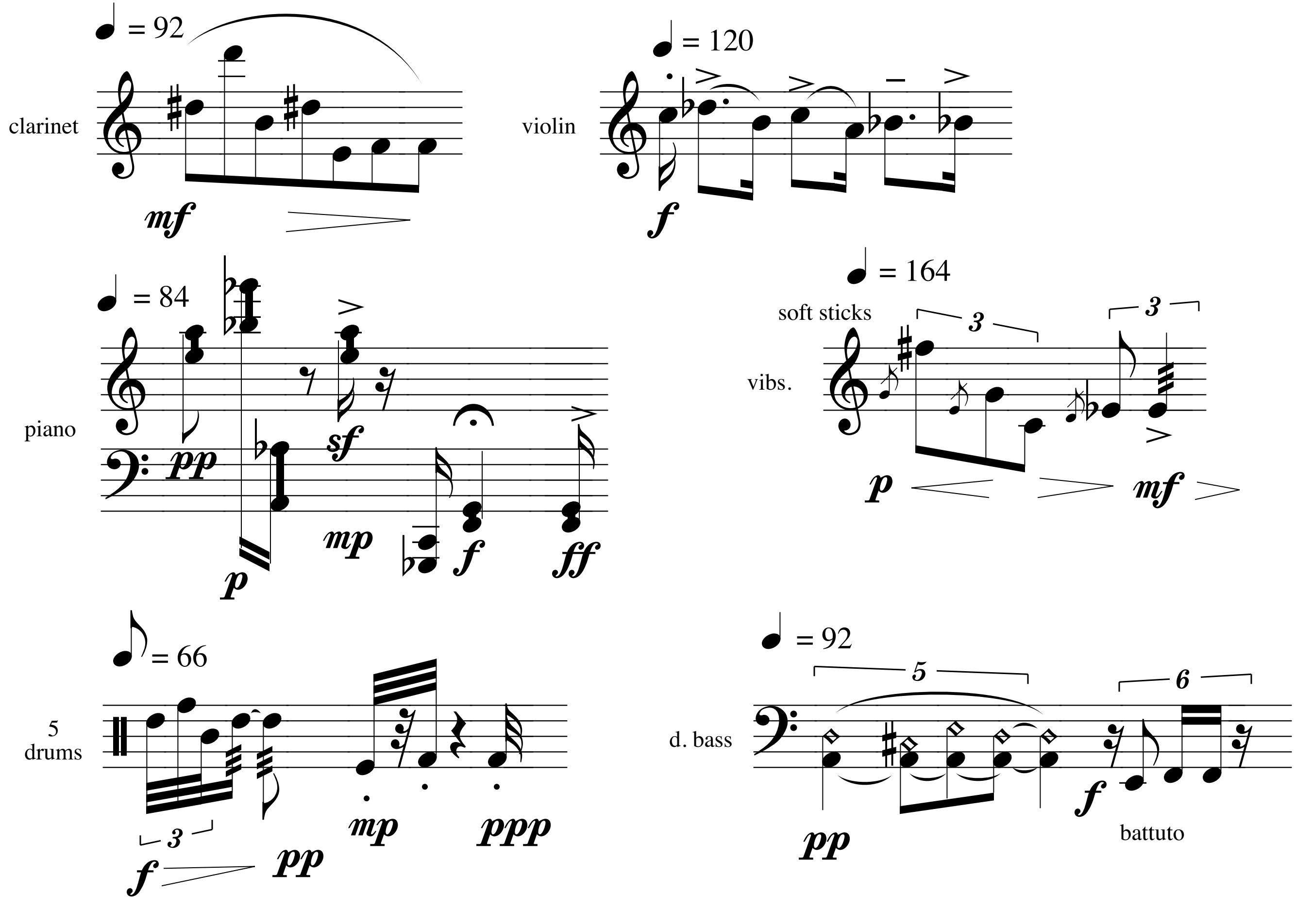
[64] "B" is the pc 11.

[65] The first order position of a pcseg is 0, the next 1, etc.

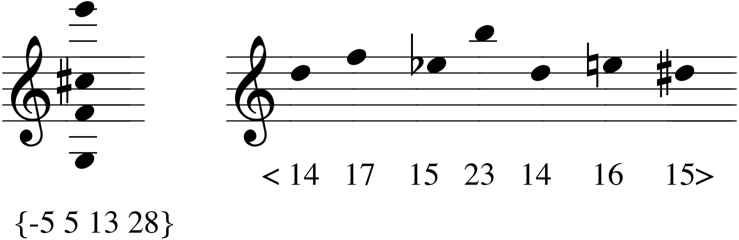
[66] We say they partition the order positions of S.

[67] An involution is a transformation that is its own inverse. Doing it twice is the same as doing nothing. So if F is an involution, F^2 = identity. F^2 is another way of writing FF. Incidentally; all T_nI operators are involutions (for any n).

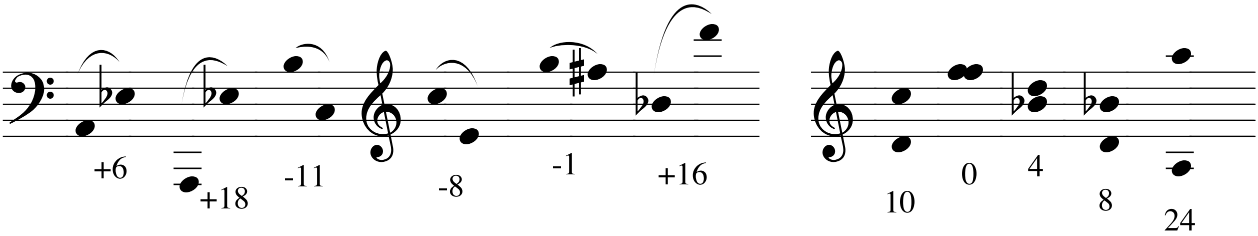
Example 1 Six instances of the contour <3,4,2,3,0,1,1>



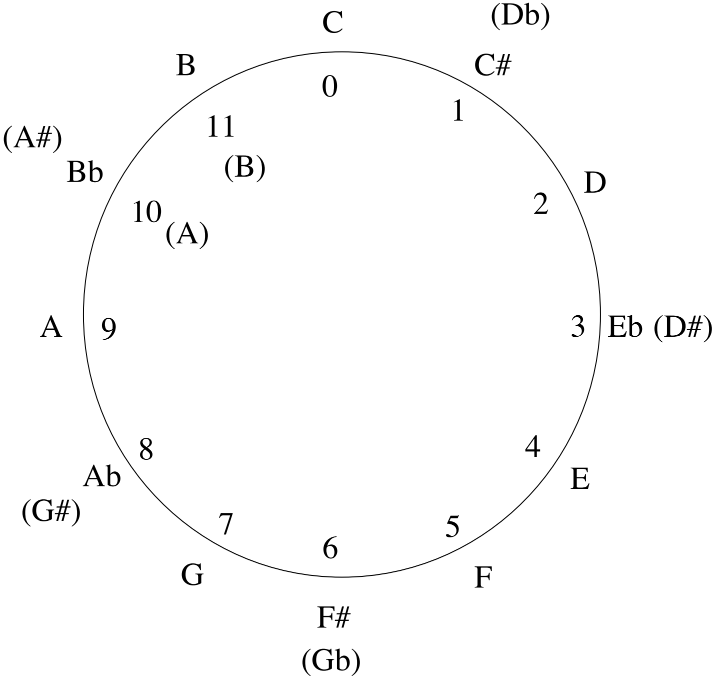
Example 2 Pitch space chord (unordered set and melody (ordered set))



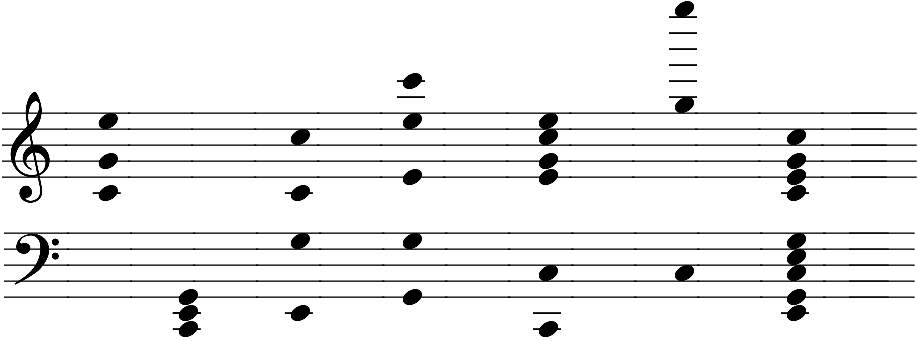
Example 3 Pitch-space unrdered and ordered intervals



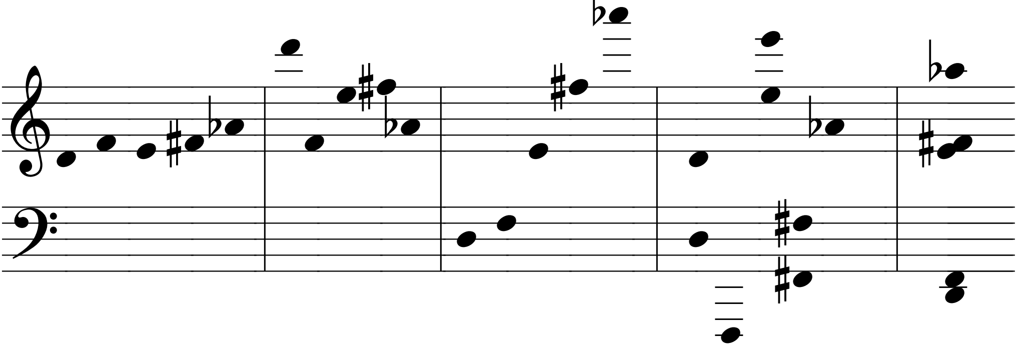
Example 4 Pitch-class space



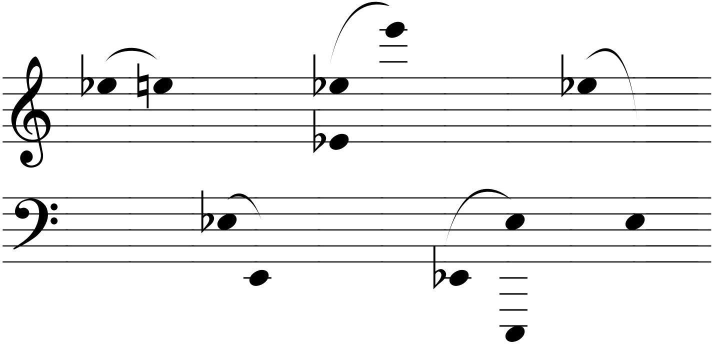
Example 4a.
The pitch-class circle with
notation for each pitch-class (pc).



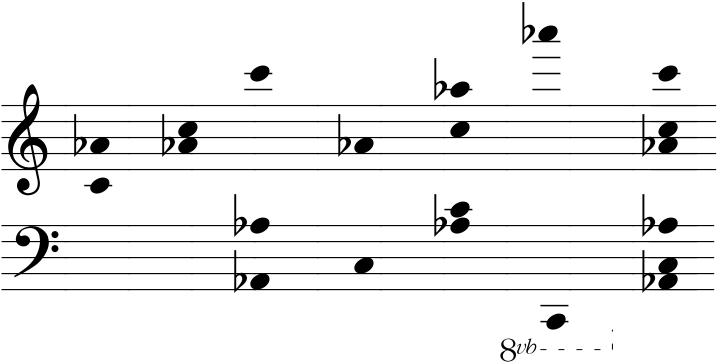
Example 4b.
Some realizations of the C-major chord {C E G} or {047}
in the pitch-space.



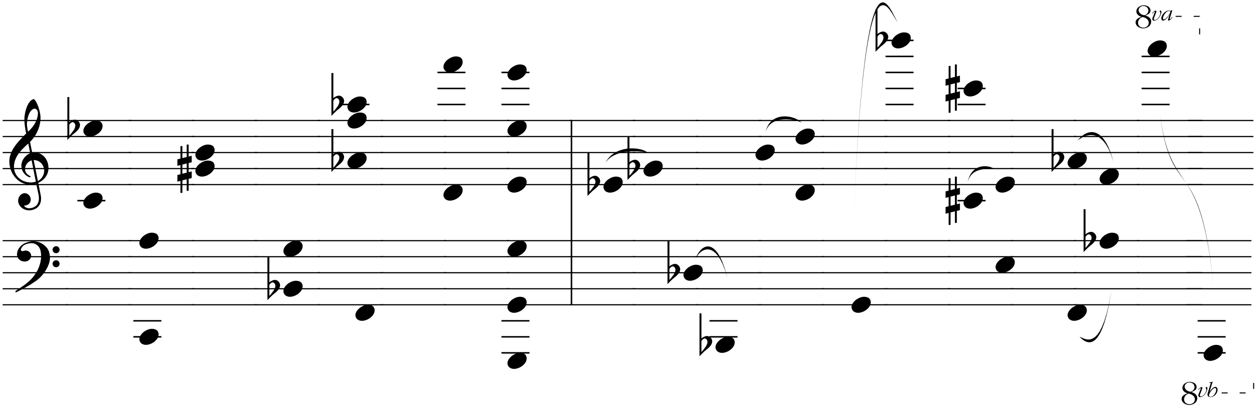
Example 4c.
Some realizations of a series of pcs <D F E F# Ab> or <25468>
in the pitch-space.



Example 4d.
Some realizations of the ordered pc-space interval
<Eb E> or <34> of pcs .



Example 4e.
Some realizations of the unordered pc-space interval
{C Ab} or {08} of pcs.

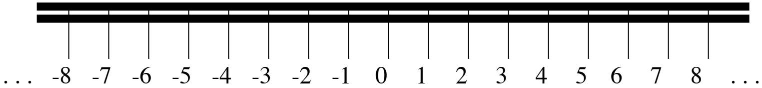


Some realizations of the unordered pc-space interval 3 in the pitch space.

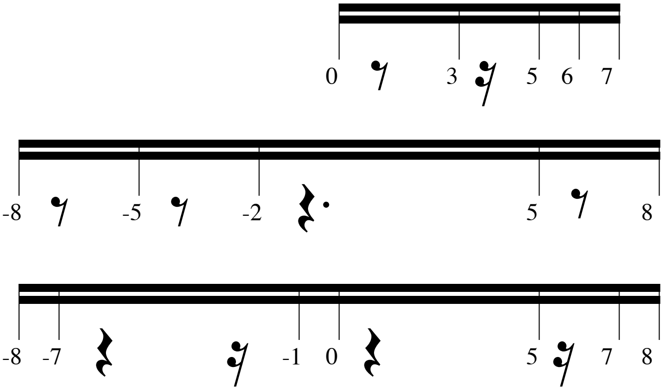
Some realizations of the ordered pc-space interval 3 in the pitch space.

Example 4f.

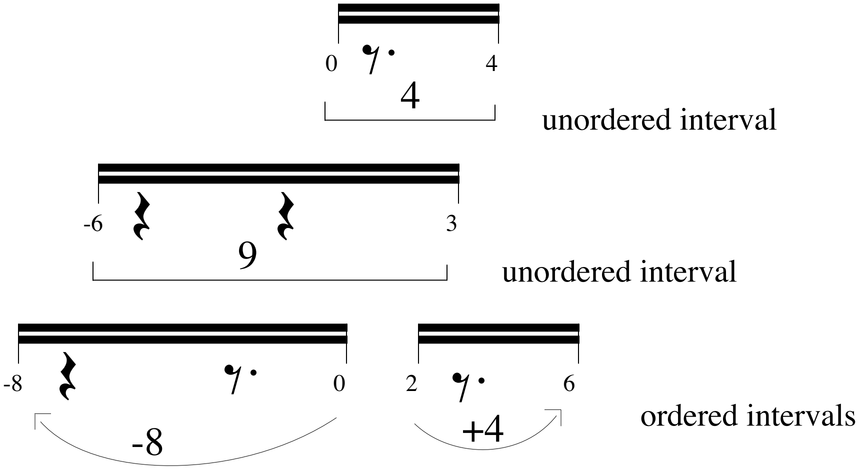
Example 5 measured time space



Example 5a
A measured time space shown as
a series of sixteeth notes.

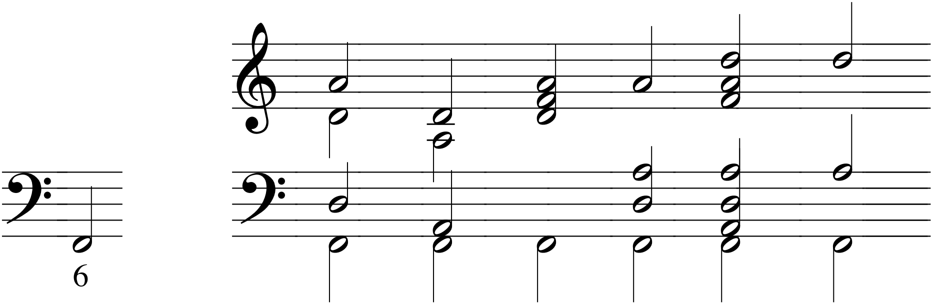


Example 5b
Rhythms as subsets selected out of the
measured time space of Example 5a.



Example 5c
unordered intervals (durations) between time points
and ordered intervals from one time point to another
in the measured time space of Example 5a.

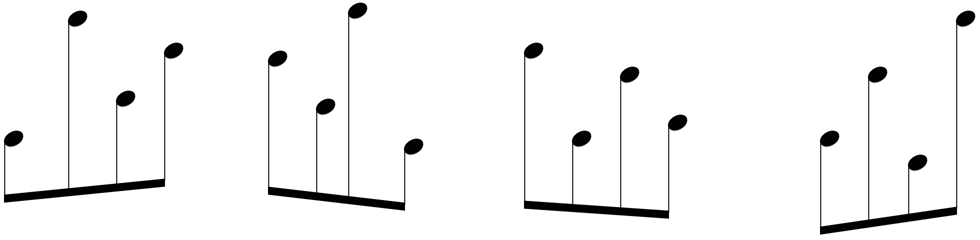
Example 6 Partially ordered sets model figured bass.



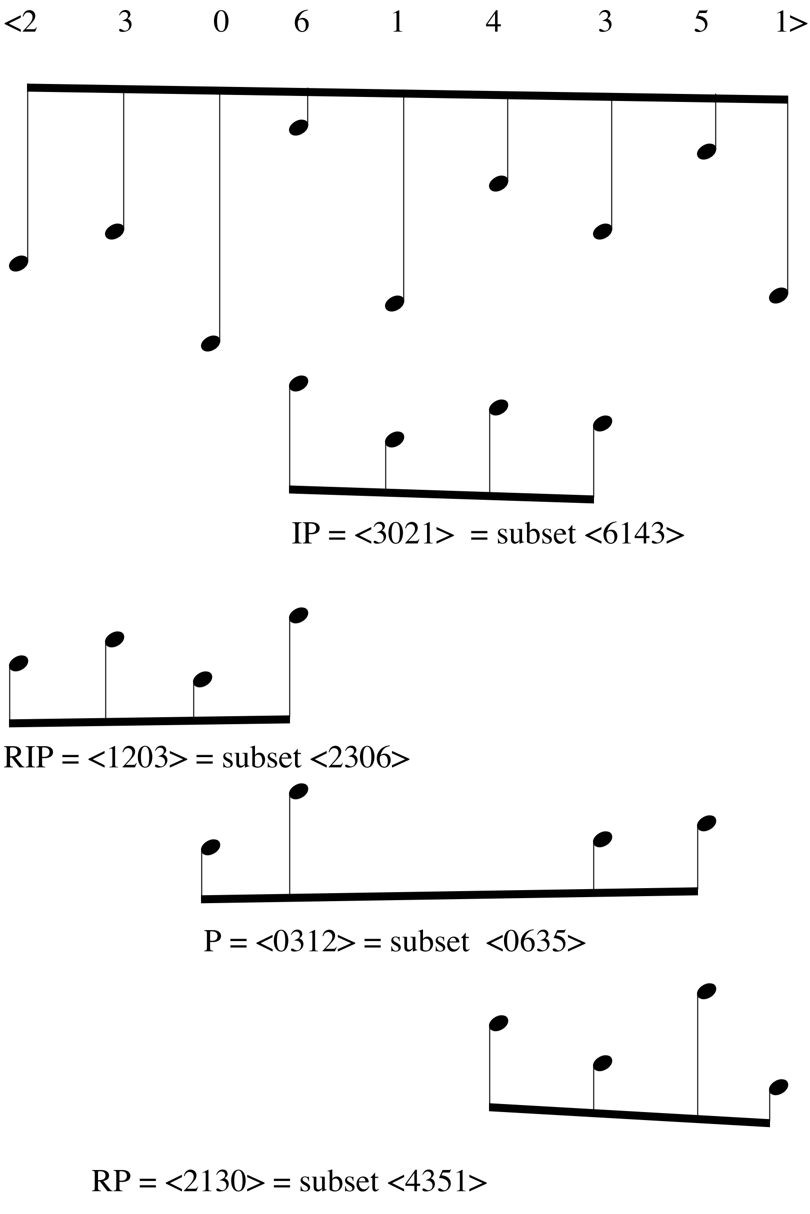
Figured bass

Various realizations of
partially ordered pc set < F {D F A} > or <5{259}>
in pitch (from low to high).

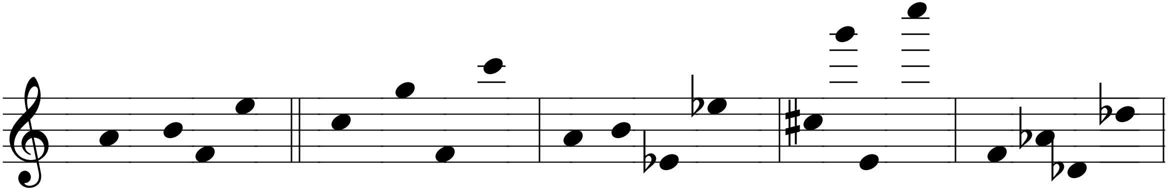
Example 7 A contour-class including contour P = <0312>



Example 8 A supercontour including the members of the contour-class of Example 7



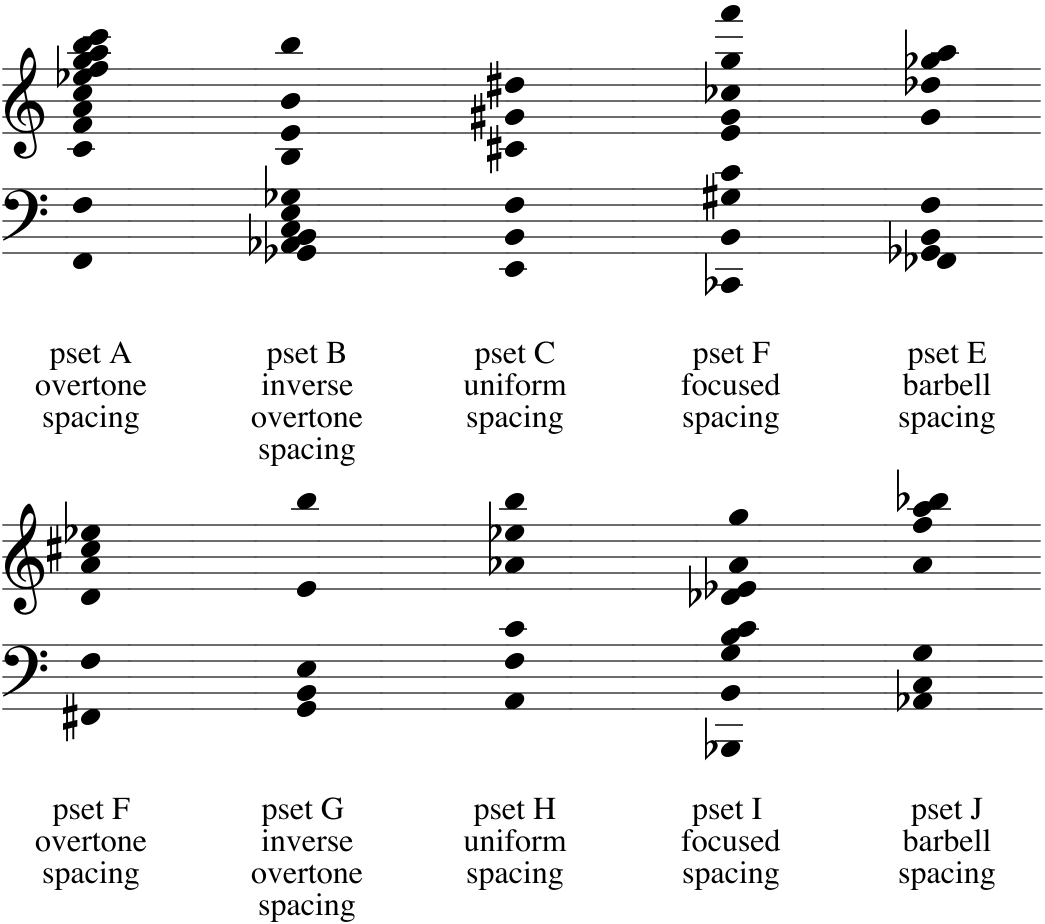
Example 9 Contour expansion in pitch-space.



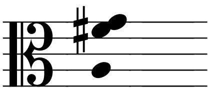
S = <9 11 5 16>   contour expansion of S:

To expand the contour of a ordered pitch space set, increase the size of any of the intervals between the adjacent pitches so that the resulting ordered pitch set is an equivalent contour (in contour space).

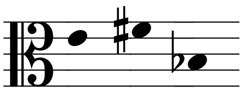
Example 10 Spacing types of unordered pitch space sets (psets)



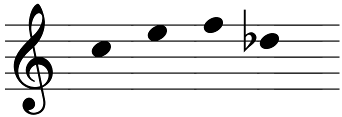
Example 11 Transposition of sets in pitch-space



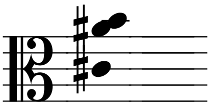
pset A = {-3 6 7}



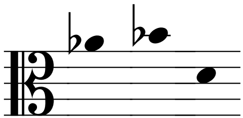
pseg B = <4 6 -2>



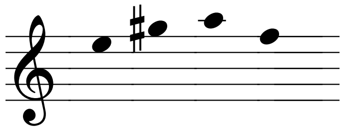
pseg C = <12 16 17 13>



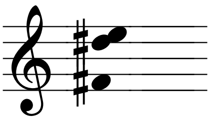
T_4A = {1 10 11}



T_4B = <8 10 2>

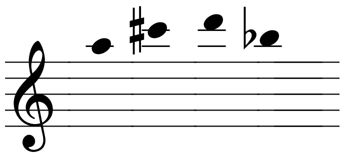


T_4C = <16 20 21 17>

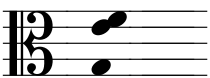


T_9A = {6 15 16}

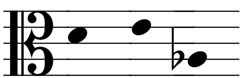
T_9B = <13 15 7>



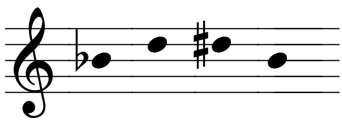
T_9C = <21 25 26 22>



$T_{-2}A$ = {-5 4 5}

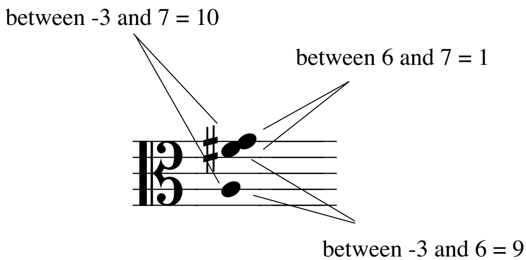


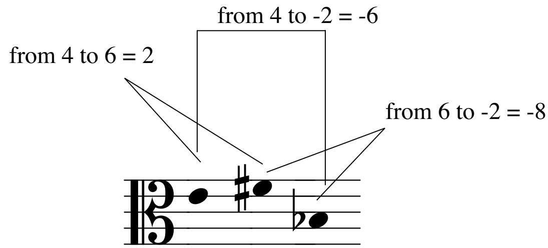
$T_{-2}B$ = <2 4 -4>



$T_{-2}C$ = <10 14 15 11>

Example 12 Intervals between pitches of sets in pitch-space





pset A = {-3 6 7}

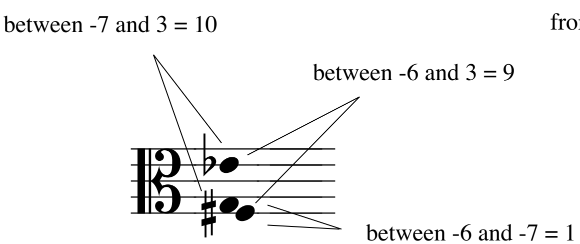
unordered intervals: 1 9 10

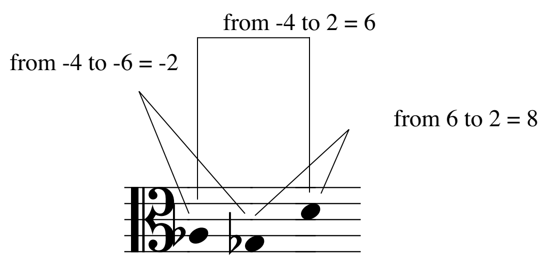
pseg B = <4 6 -2>

ordered intervals: -8 -6 2

Transposition preserves both ordered and unordered intervals.

Inversion preserves unordered intervals, but inverts ordered intervals. See below:





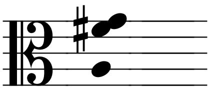
pset IA = {3 -6 -7}

unordered intervals: 1 9 10

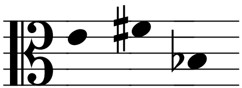
pseg IB = <-4 -6 2>

ordered intervals: -2 6 8

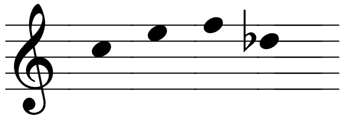
Example 13 Transposed inversions of sets in pitch-space



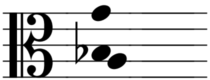
pset A = {-3 6 7}



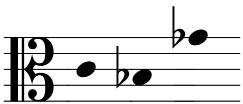
pseg B = <4 6 -2>



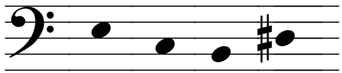
pseg C = <12 16 17 13>



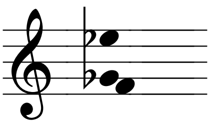
T_4IA = {7 -2 -3}



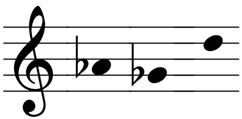
T_4IB = <0 -2 6>



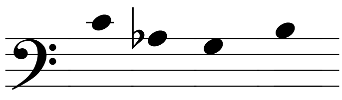
T_4IC = <-8 -12 -13 -9>



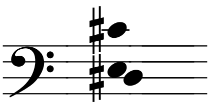
T_{12}IA = {15 6 5}



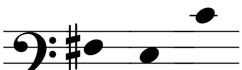
T_{12}IB = <8 6 14>



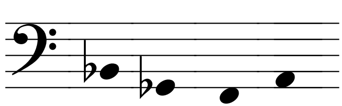
T_{12}IC = <0 -4 -5 -1>



T_{-2}IA = {1 -8 -9}

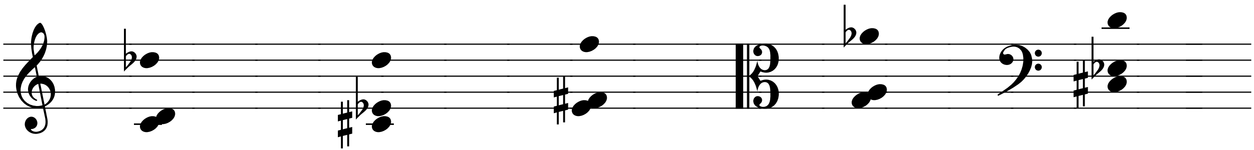


T_{-2}IB = <-6 -8 0>

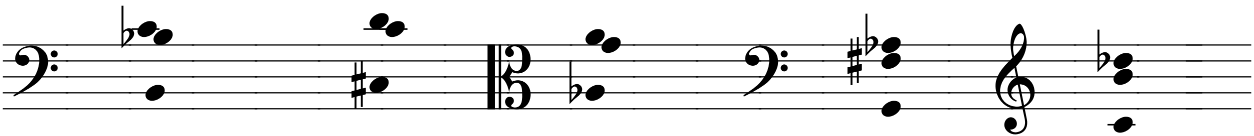


T_{-2}IC = <-14 -18 -19 -15>

Example 14 Members of the pitch-space set-class SC(K) where K = {0 2 13}

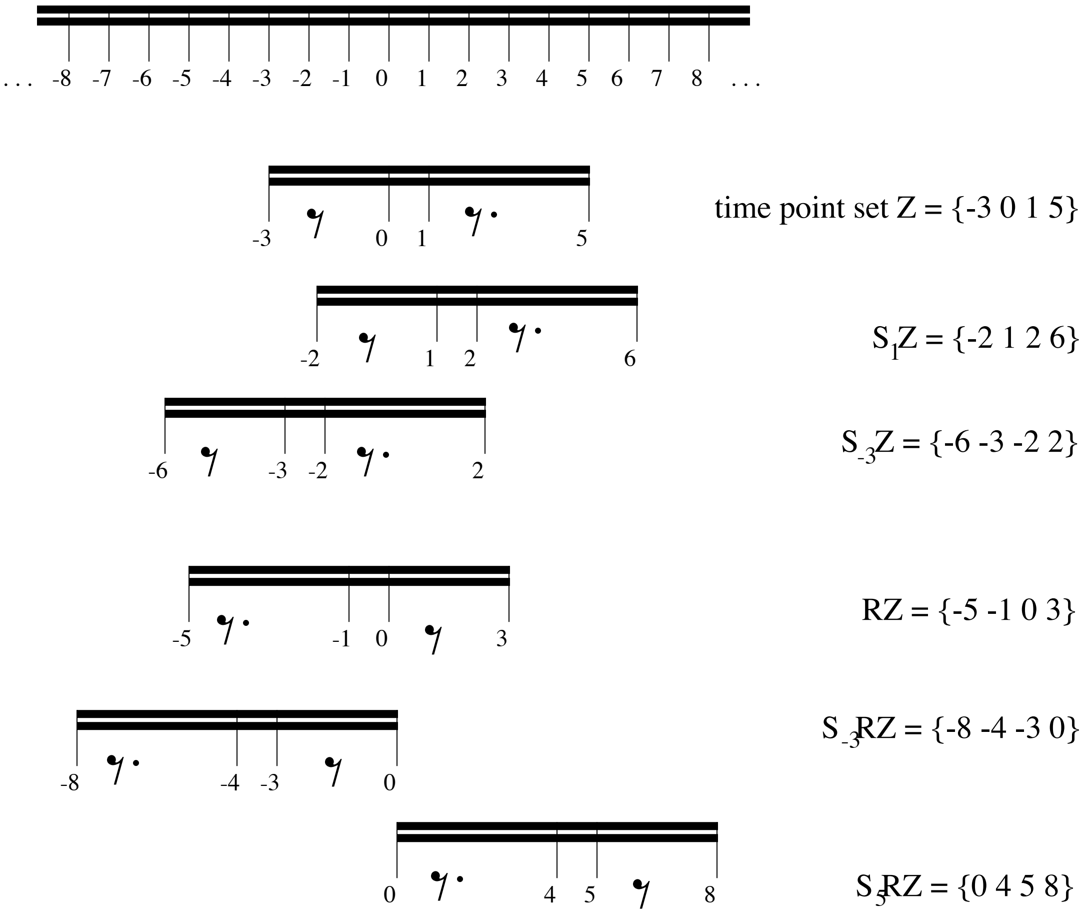


K = {0 2 13} T_1K = {1 3 14} T_4K = {4 6 17} T_{-5}K = {-5 -3 8} T_{-11}K = {-11 -9 2}



IK = {0 -2 -13} T_2IK = {2 0 -11} T_9IK = {9 7 -4} T_{-4}IK = {-4 -6 -17} T_{13}IK = {13 11 0}

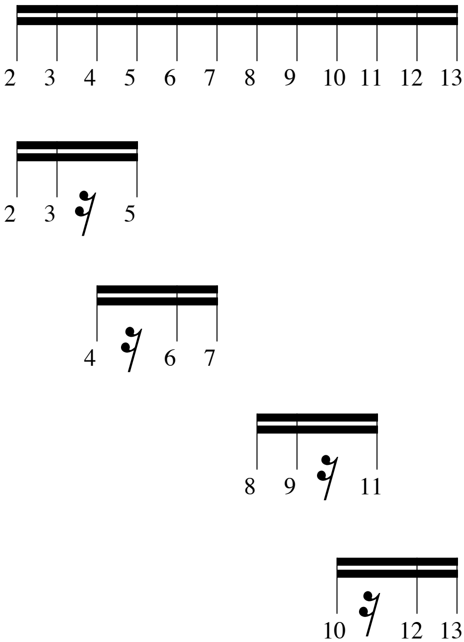
Example 15 Shift (S_n) and retrograde (R) among time point sets



The unordered intervals within Z and S_nZ and S_nRZ are
3, 4, 8, 1, 5, 4.

Example 16

An unordered time point set is partitioned into members of a time point set-class



unordered time point set Q = {2 3 4 5 6 7 8 9 10 11 12 13}

unordered time point set P = {2 3 5}

S_9RP = {4 6 7}

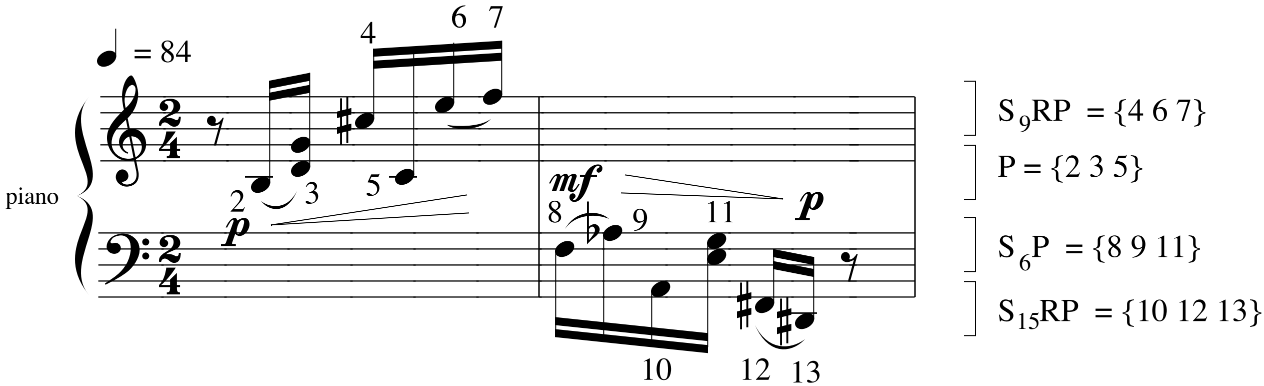
S_6P = {8 9 11}

$S_{15}RP$ = {10 12 13}

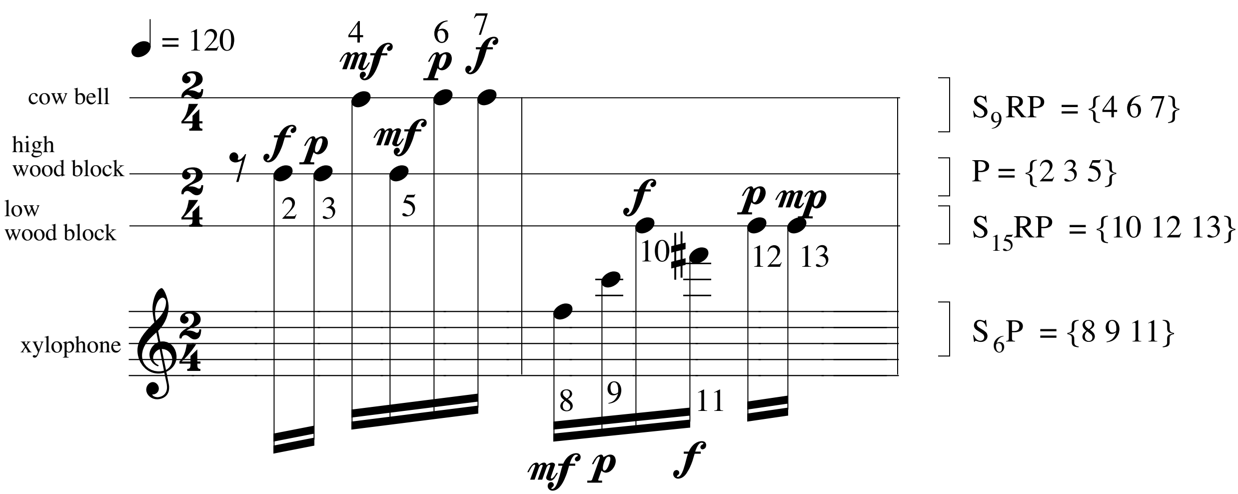
members of SC(P)

Example 17 Musical realizations of Example 16.

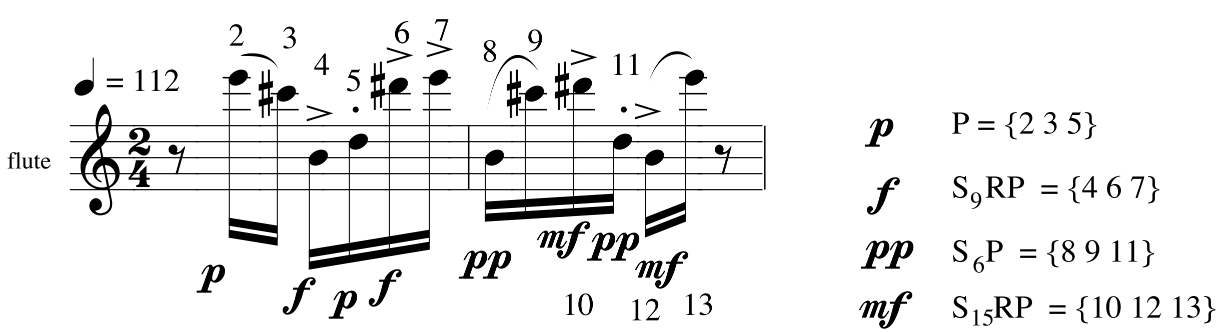
Members of SC(P) articulated by register (in pitch space)



Members of SC(P) articulated by timbre.



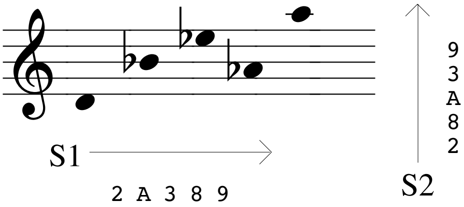
Members of SC(P) articulated by dynamics.



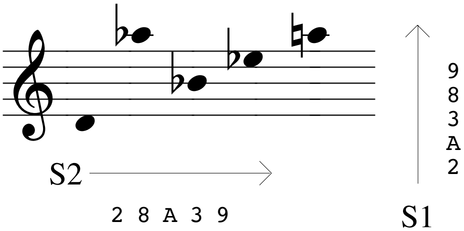
Example 18 A set class set ordered in (both) pitch and time.

pcset S = {2389A}
S1 = an ordering of S = <2A389>
S2 = another ordering of S = <28A39>

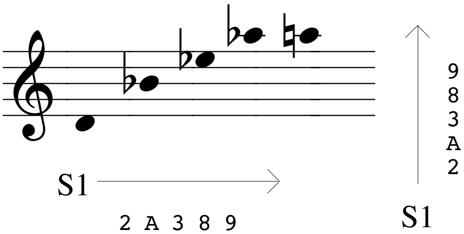
NB:
pc A = pc 10



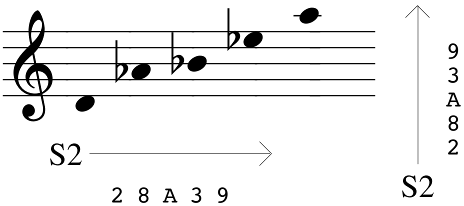
S is realized in time by S1 and in pitch by S2.



S is realized in time by S2 and in pitch by S1.

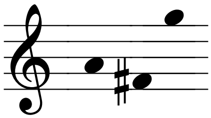


S is realized in time by S1 and in pitch by S1.

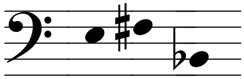


S is realized in time by S2 and in pitch by S2.

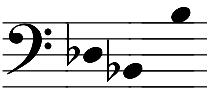
Example 19 Realizations of pcset A and pcseg B and their pc transpositions.



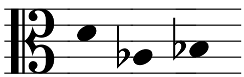
pset A = {9 6 7}



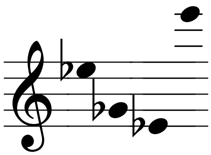
pseg B = <4 6 A>;
the ordering applies
to time.



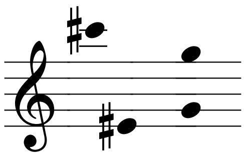
T_4A = {1 A B}



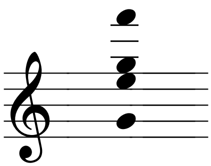
T_4B = <8 A 2>;
the ordering applies
to pitch.



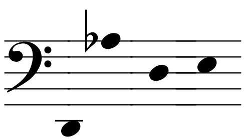
T_9A = {6 3 4}



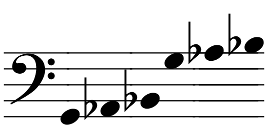
T_9B = <1 3 7>;
the ordering applies
to time.



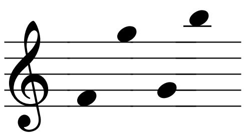
T_AA = {7 4 5}



T_AB = <2 4 8>;
the ordering applies
to pitch.



T_1A = {A 7 8}



T_1B = <5 7 B>;
the ordering applies
to time.

NB:
pc A = pc 10
pc B = pc 11

In this example A and B are also the names of pitch-class space sets.

Remember that each realization of a transposition of pcset A has a vertical (pitch) ordering and a horizontal (time) ordering. In this example some of these orderings involve duplications.

The realizations of transpositions of pcseg B are ordered, but the ordering may apply only to either pitch or time with the other ordering being free (possibly with duplications).

Example 20 Table of pcs and their inverses (or inversions)

a	Ia
0	0
1	B
2	A
3	9
4	8
5	7
6	6
7	5
8	4
9	3
A	2
B	1

Remarks

I is the inversion operator.

a is a pc or interval.

I(Ia)) = a (The inverse of the inverse of a is a).

Ia + a = 12

The pair a and Ia produces 7 distinct pairs: 0 and itself; 1 and B; 2 and A; 3 and 9; 4 and 8; 5 and 7; and 6 and itself.

Example 21 Intervals between pcs of sets in pitch-class space

NB:
pc A = pc 10
pc B = pc 11

In this example A and B are also the names of pitch-class space sets.

between 9 and 6 =3 — between 6 and 7 = 1

pset A = {9 6 7}

between 9 and 7 = 2

unordered intervals: 1 2 3

from 4 to 6 = 2 — from 6 to A = 4

pseg B = <4 6 A>

from 4 to A = 6

ordered intervals: 2 4 6

Transposition preserves both ordered and unordered intervals.

Inversion preserves unordered intervals, but inverts ordered intervals.
See below:

between 3 and 6 =3 — between 6 and 5 = 1

pset IA = {3 6 5}

between 3 and 5 = 2

unordered intervals: 1 2 3

from 8 to 6 = A — from 6 to 2 = 8

pseg IB = <8 6 2>

from 8 to 2 = 6

ordered intervals: 6 8 A

Example 22 T_nI transformations on pcsets and pcsegs.

NB:
pc A = pc 10
pc B = pc 11

In this example A and B are also the names of pitch-class space sets.

Pset A = {967}

T_4IA = {7A9}

T_9IA = {032}

T_2IA = {587}

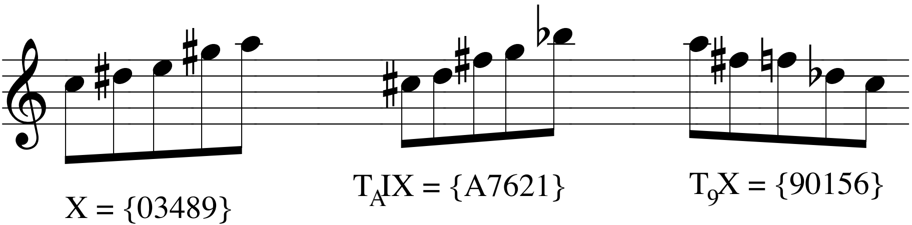
Pcseg B = <46A>

T_4IB = <0A6>

T_9IB = <53B>

T_2IB = <A84>

Example 23 Three pcsets related by T_n and T_nI.



Example 24 The pc set-classes SC(3-2) and SC(6-20)

SC(3-2) = SC({013}); 24 members; ICV[3111000]

{013}	=X	{0B9}	=IX
{124}	=T_1X	{10A}	=T_1IX
{235}	=T_2X	{21B}	=T_2IX
{346}	=T_3X	{320}	=T_3IX
{457}	=T_4X	{431}	=T_4IX
{568}	=T_5X	{542}	=T_5IX
{679}	=T_6X	{653}	=T_6IX
{78A}	=T_7X	{764}	=T_7IX
{89B}	=T_8X	{875}	=T_8IX
{9A0}	=T_9X	{986}	=T_9IX
{AB1}	=T_AX	{A97}	=T_AIX
{B02}	=T_BX	{BA8}	=T_BIX

SC(6-20) = SC({014589}); 4 members; ICV[6303630]

{014589}	=X = T_4X = T_8X = T_1IX = T_5IX = T_9IX
{12569A}	=T_1X = T_5X = T_9X = T_1IX = T_6IX = T_AIX
{2367AB}	=T_2X = T_6X = T_AX = T_2IX = T_7IX = T_BIX
{3478B0}	=T_3X = T_7X = T_BX = T_3IX = T_8IX = T_0IX

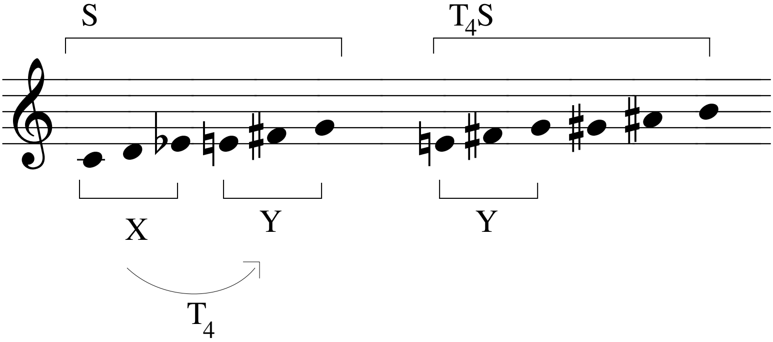
Example 25 Set-classes in pc space.

	cardinality	# of pcsets	# of set-classes
	0	1 (Ø)	1
	1	12 (pcs)	1
	2	66	6
	3	220	12
	4	495	29
	5	792	38
	6	924	50
	7	792	38
	8	495	29
	9	220	12
	10	66	6
	11	12	1
	12	1 (U)	1
total		4096	224

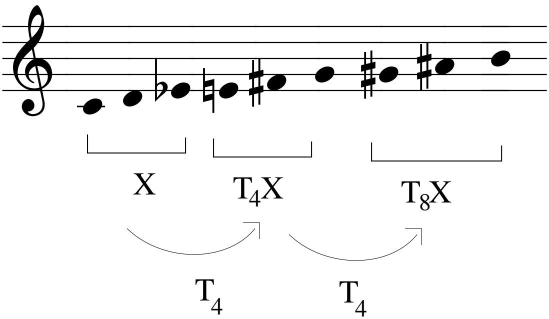
Example 26 Invariance in pitch space.

S = <0 2 3 4 6 7>

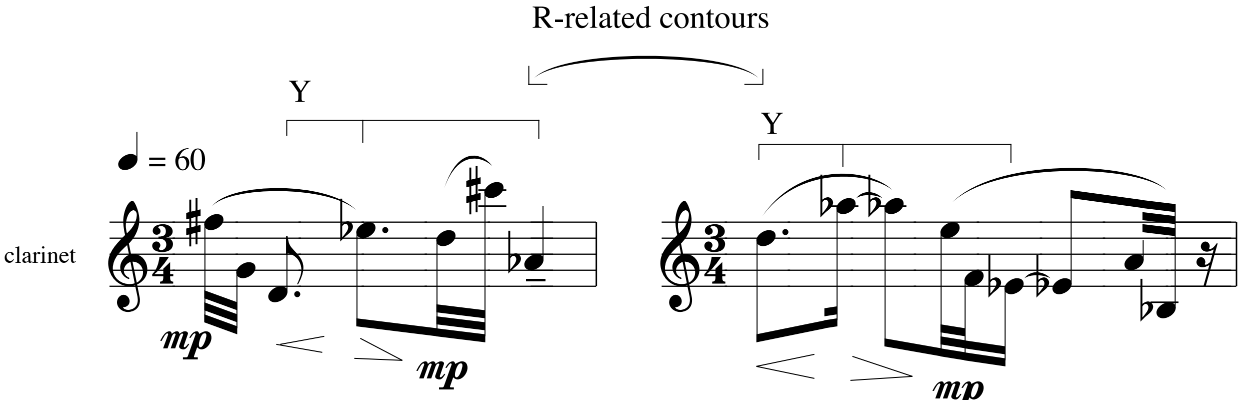
X = <0 2 3>   Y = <4 6 7> = T_4X



One can link S and T_4S to produce a nine-pitch pseg "generated" by successive T_4 transpositions of X.



Example 27 Invariance in pc space realized in pitch space



S = {0123678}

T_2S = {234589A}

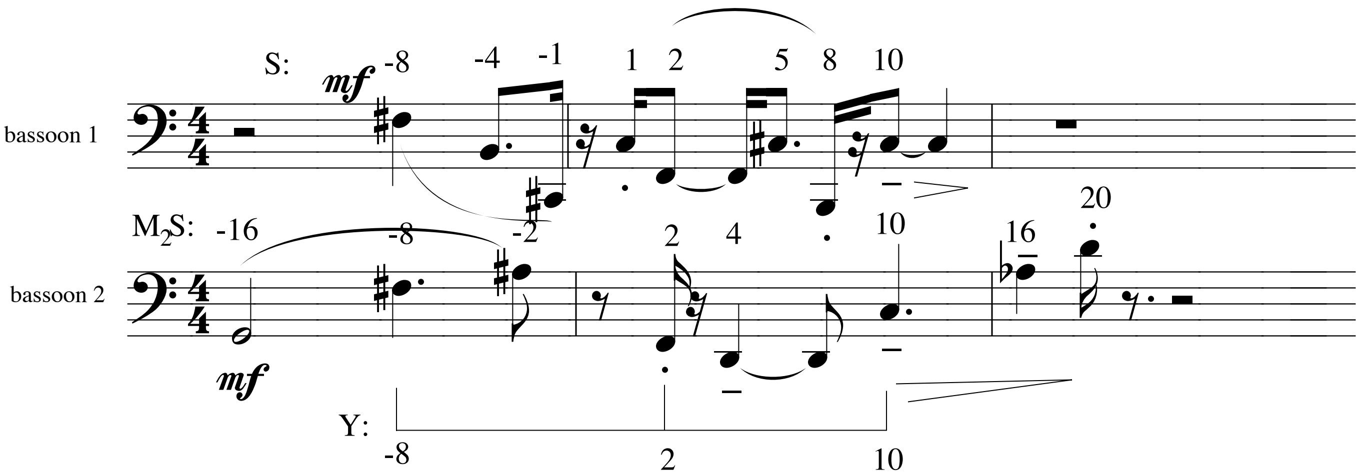
Y= {238}

Example 28 Multiplicative invariance in measured time space

S = {-8 -4 -1 1 2 5 8 10}

M_2S = {-16 -8 -2 2 4 10 16 20

Y = {-8 2 10}



The time points of Y share the same pitches.

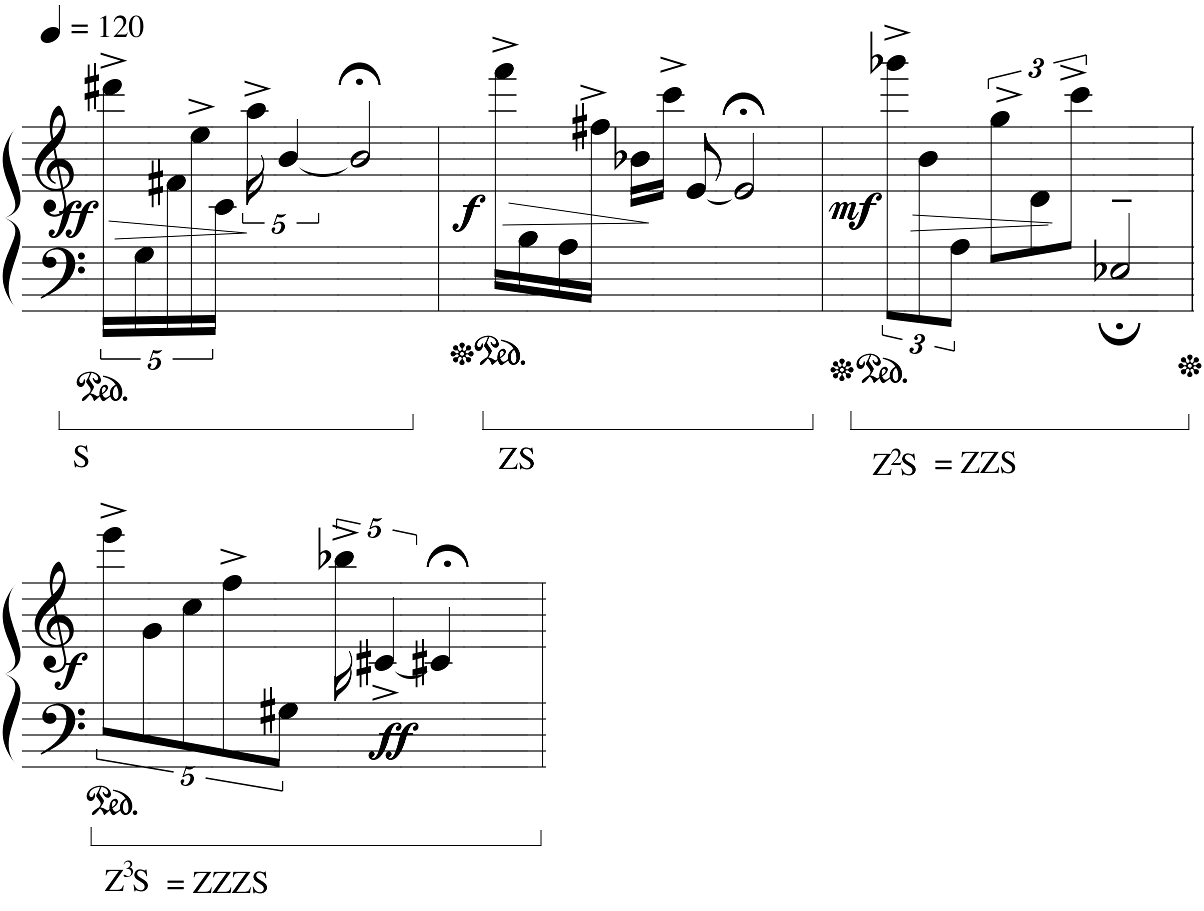
Example 29 Invariance in contour space

Permutation Z on contour S preserve ordered and position of Z's three highest pitches.
S = <6024153>

perrmutation Z:

0 1 2 3 4 5 6
1 3 0 2 4 5 6

cycles of Z: (0132)(4)(5)(6)



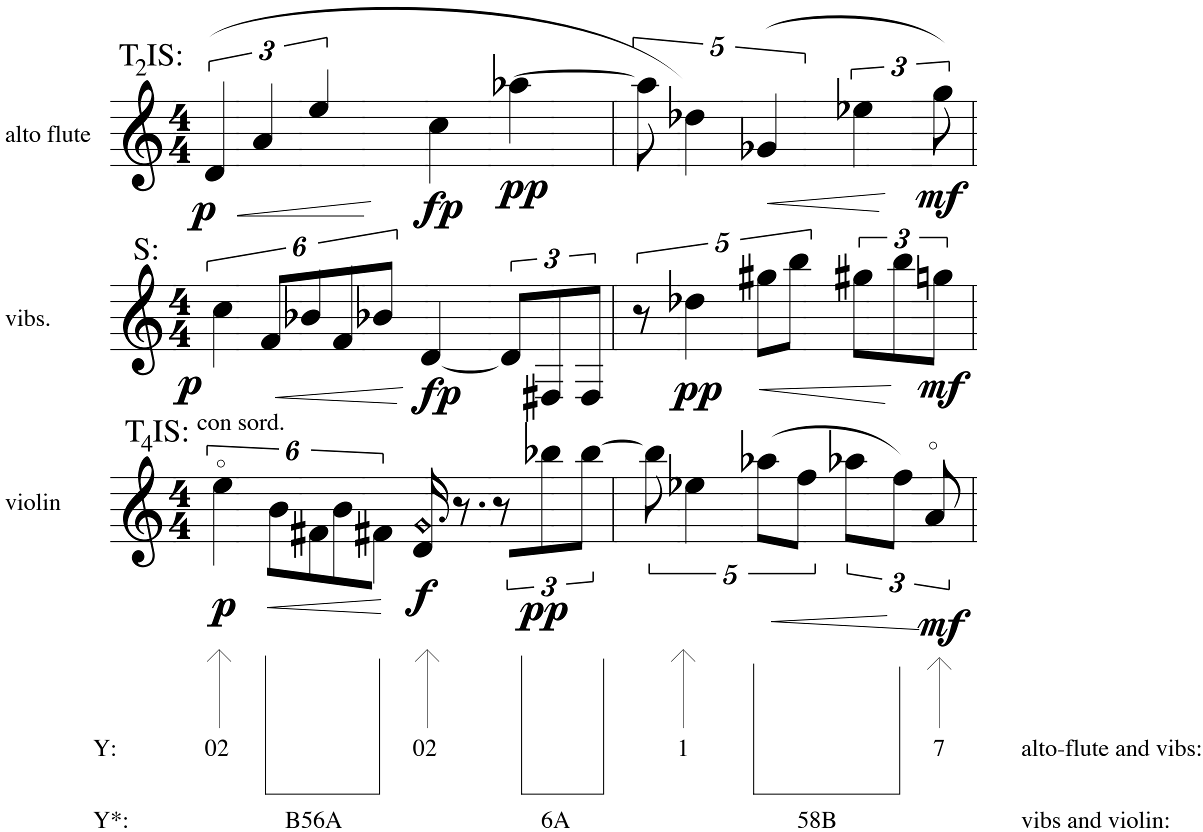
Example 30 Invairance in pc space among three psegs.

T_2IS: <2 9 4 0 8 1 6 3 7>
S: <0 5 A 2 6 1 8 B 7>
T_4IS <4 B 6 2 A 3 8 5 9>

Y = {0127}
Y* = {568AB}

T_2IS and S share the pcs of Y in the same order positions.

T_4IS and S share the pcs of Y* in the same order positions.



Example 31 Invariance involving the exchange of sets in pc space

$S = \langle 02346 \rangle$

$RT_6IS = \langle 20346 \rangle$

$X = \langle 034 \rangle$

$Y = \langle 236 \rangle$