The Complement Theorem

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In “Bob’s Atonal Theory Primer,” we stated the complement theorem as follows:

**Complement Theorem**: For pcsets A and A’, the ICV of A’ is a transformation of the ICV of A: for each entry in the ICV of A except the entry for ic6, add k (for the entry of ic6, add k/2). \( k = 2a – 12 \), where a is the number of elements in A.

(NB: \( k = a – (12-a) \)).

**Corollary (the Hexachord Theorem)**: If A is a hexachord, then \( k = 0 \), so complementary hexachords have the same ICV.

We will prove the theorem in an abstract way and interpret the result so it applies to pcs and intervals.

Consider example 1a: 12 points are placed in a space divided into two sectors 1 and 2 separated by a boundary. The number of points in sector one is \( P_1 \); the number of points in sector two is \( P_2 \).

\[ (1) \quad 12 = P_1 + P_2 \quad \text{(and } P_2 = 12 – P_1) \]

Now let us connect all the points with arrows so that each point \( p_n \) is connected to exactly two other points such that \( p_n \) is connected by an arrow to \( p_m \) and \( p_l \) is connected to \( p_n \) by an arrow for all points \( p_n, p_m, p_l \). See example 1b. \( p_m \) may equal \( p_l \) as in example 1c. Examples 1d to 1e show the points of 1a connected by arrows in two different ways. The arrows connect the points into one or more cycles. In each of these examples \( n_1 \) is the number of arrows that connect the points in sector 1; \( n_2 \) is the number of arrows connecting points in sector 2; \( n_0 \) is the number of arrows that connect points from sector 1 to 2 or sector 2 to 1.

\[ (2) \quad 12 = n_1 + n_2 + n_0. \]

Now examine example 2. In example 2a two arrows connect four points; in sector 1 \( p_1 \) connects to \( p_2 \) and in sector 2 \( p_3 \) connects to \( p_4 \). The arrows do not cross the boundary between sector 1 and 2. In example 2b, the same four points are reconnected so that arrows cross the boundary: \( p_1 \) connects to \( p_3 \) and \( p_4 \) to \( p_2 \). This changes the values of \( n_1, n_2 \) and \( n_0 \) without changing \( P_1 \) and \( P_2 \); both \( n_1 \) and \( n_2 \) are diminished by 1 and \( n_0 \) is increased by 2.

\[ \text{(3) In the entire space of 12 points and arrows, as we reconnect pairs of points as indicated above, for each pair of points } n_0 \text{ increases by 2 while } n_1 \text{ and } n_2 \text{ both decrease by 1 and } P_1 \text{ and } P_2 \text{ remain invariant.} \]

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1 It should be clear that we must change *pairs* of arrows; otherwise some points will either be without arrows pointing to or away from them, or have more than one arrow pointing to or away.
Now imagine the case where there are no points connected across the boundary between sectors 1 and 2; that means $n_0 = 0$. Ex. 3 shows such a situation. So

(4) if $n_0 = 0$, then $P_1 = n_1$ and $P_2 = n_2$.

Let (4) be the case. When we change $n_0$ to $x$ by reconnecting points, from (3) we have

(5) $n_1 = P_1 - x/2$ and $n_2 = P_2 - x/2$.

(Note that $x$ has to be even.)

From (2) and (5) we derive:

(6) $12 = n_1 + (P_2 - n_0/2) + n_0$.

We solve (6) for $n_0$.

From (1), $12 = n_1 + (12 - P_1 - n_0/2) + n_0$

$P_1 = n_1 + n_0/2$

(7) $n_0 = 2(P_1 - n_1)$

Now from (2) and (7) we have

$12 = n_1 + n_2 + 2(P_1 - n_1)$.

We solve for $n_2$.

$12 = n_1 + n_2 + 2P_1 - 2n_1$

$n_2 = 12 - 2P_1 + n_1$. Now we rearrange terms.

(8) $12 - 2P_1 = n_2 - n_1$

From (1) we write

$P_1 + P_2 - 2P_1 = n_2 - n_1$

and derive the result:

(9) $P_2 - P_1 = n_2 - n_1$.

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2 Compare Exx. 1d, 1e and Ex. 3 for an example of the change of arrows among the same points and the values of $n_0$, $n_1$, and $n_2$.
(9) shows that the difference between $P_1$ and $P_2$ is the difference between the number of arrows that connect points in sector 1 and the number of arrows that connect points in sector 2. Note that if $P_1 = P_2$, then $n_1 = n_2$.

We may reinterpret (9) so that the points are pcs, and redefine $P_1$ and $P_2$ as complementary sets. When we interpret the arrows as directed intervals for each directed interval $n$, the resulting cycles of pcs form the cycles of interval $n$. Thus $n_1$ is the number of intervals $n$ within the pcset $P_1$; $n_2$ = the number of intervals $n$ within pcset $P_2$, and $n_0$ is the number of intervals $n$ between pcsets $P_1$ and $P_2$. This means that (9) can be reinterpreted as

(10a) The difference between the cardinalities of complementary pcsets $P_1$ and $P_2$ is equal to the difference between number of intervals $n$ within $P_1$ and the intervals $n$ within $P_2$.

(10b) If $P_1$ and $P_2$ are complementary hexachords, there is no difference between the number of intervals $n$ within $P_1$ and the number of intervals $n$ within $P_2$.

Let $IF(Z,Z)$ be the interval function of pcset $Z$ and itself\(^3\) so we can generalize (10a and b).

(11a) Complement Theorem: For complementary pcsets $P_1$ and $P_2$, the $IF(P_2,P_2)$ may be derived from $IF(P_1,P_1)$ by adding $k$ to each argument in $IF(P_1,P_1)$, where $k$ is the difference between the cardinalities of $P_1$ and $P_2$.

(11b) Hexachord Theorem: If $P_1$ and $P_2$ are hexachords, then $IF(P_2,P_2) = IF(P_1,P_1)$.

(11a,b) can be applied to the interval class vectors (ICV) of $P_1$ and $P_2$, but the value of the last argument of the vectors has to be divided by 2.

Examples:

Let $P_1 = \{013467\}$ and $P_2 = \{2589AB\}; k = 0$; $IF(P_1,P_1) = IF(P_2,P_2) = [63242242423]$; $ICV(P_1) = ICV(P_2) = [632422]$.

Let $P_1 = \{0236\}$ and $P_2 = \{146789AB\}; k = 4$; $IF(P_1,P_2) = [411210201211]$ and $IF(P_2,P_2) = [85565456565] ; ICV(P_1) = [4112101]$ and $ICV(P_2) = [8556543]$.

\(^3\) The interval function is a 12-argument array that lists the intervals between two pcsets, $X$ and $Y$; it is written $IF(X,Y)$. The $n$th argument in the array gives the number of directed intervals of size $n$ from $X$ to $Y$. For example, let $X = \{024\}$ and $Y = \{235\}$. $IF(X,Y) = [121201000011]$; argument 0 is 1, indicating the one common pc between $X$ and $Y$; argument 1 is 2, indicates the two interval 1s from 2 in $X$ to 3 in $Y$ and 4 in $X$ to 5 in $Y$. The $ICV(X)$ and $IF(X,X)$ provide similar information—except in the case of interval 6, as indicated in the text—but $ICV(X)$ gives the number of interval-classes within $X$, where the $IF(X,X)$ gives the number of directed intervals between $X$ and a copy of $X$. 

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Ex. 1a

sector 1

\[ P_1 = 5 \]

Ex. 1b

sector 2

\[ P_2 = 7 \]

Ex. 1c

\[ p_n \rightleftharpoons p_1 = p_m \]

Ex. 1d

\[ P_1 = 5 \]

\[ P_2 = 7 \]

\[ n_1 = 4 \quad n_0 = 2 \quad n_2 = 6 \]

Ex. 1e

\[ P_1 = 5 \]

\[ P_2 = 7 \]

\[ n_1 = 3 \quad n_0 = 4 \quad n_2 = 5 \]

Ex. 2a

\[ p_1 \leftarrow p_2 \]

\[ p_3 \leftarrow p_4 \]

Ex. 2b

\[ p_1 \rightarrow p_2 \]

\[ p_3 \rightarrow p_4 \]

Ex. 3

\[ P_1 = 5 \]

\[ P_2 = 7 \]

\[ n_1 = 5 \quad n_0 = 0 \quad n_2 = 7 \]